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Well-posedness for stochastic Camassa–Holm equation

Yong Chen [a], Hongjun Gao [a,*], Boling Guo [b]

[a] Jiangsu Provincial Key Laboratory for Numerical Simulation of Large Scale Complex Systems, School of Mathematical Science, Nanjing Normal University, Nanjing 210046, China
[b] Institute of Applied Physics and Computational Mathematics, Beijing 100088, China

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ABSTRACT

The Camassa–Holm equation describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity. Due to uncertainty in the modelling and external environment, this modelling could be subject to random fluctuations. In this article, the stochastic Camassa–Holm equation with additive noise is considered. Using regularization, a local existence and uniqueness result in the Sobolev space $H^s(\mathbb{R})$ with $s > 3/2$ of stochastic Camassa–Holm equation is obtained. With the help of priori estimates, the local solution will blow up in $H^q(\mathbb{R})$ in finite time for any $q > 3/2$ when initial value satisfies some conditions.

1. Introduction

The Camassa–Holm equation

$$\partial_t u - \partial_x^2 \partial_t u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u = 0$$

was derived by Camassa and Holm in [11,16] as a model of water waves. Here u denotes the fluid velocity in the x direction or, equivalently, the height of the water's free surface above a flat bottom. In real world, the surface of fluid is non-constant pressure, or the bottom of fluid is not flat, so a forcing term has to be added to the equation. In this paper, we consider the case that the forcing term is random of white noise type, which is a very natural approach if it is assumed that the exterior

* Corresponding author.
E-mail addresses: youngchen329@126.com (Y. Chen), gaohj@njnu.edu.cn (H.-J. Gao).

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pressure is generated by a turbulent velocity field for instance. So, we study the stochastic Camassa–Holm equation as follows:

$$\partial_t u - \partial_x^2 \partial_t u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u = \Phi \frac{\partial^2 B}{\partial t \partial x}, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R},\ t > 0. \tag{1.2}$$

B is a two-parameter Brownian motion on $\mathbb{R}^+ \times \mathbb{R}$. Φ is a Hilbert–Schmidt operator. Given two separable Hilbert spaces H and $\widetilde{H}$, we denote by $L_0^2(H;\widetilde{H})$ the space of Hilbert–Schmidt operators from H into $\widetilde{H}$. When $H = L^2(\mathbb{R})$ and $\widetilde{H} = H^s(\mathbb{R})$, $L_0^2(L^2(\mathbb{R}); H^s(\mathbb{R}))$ is simply denoted by $L_2^{0,s}$. Stochastic partial differential equations and stochastic models of fluid dynamics have been the object of intense investigations, for instance, [4,12–15,28,31].

Local well-posedness for (1.1)–(1.2) without noise was discussed by Constantin [6] for the initial data in $H^s(S)$, $S = [0, 2\pi]$ with $s \geqslant 4$, and by Misiolek [26] with $s > 3/2$. Local well-posedness in the non-periodic case was proved for the initial data in $H^s(\mathbb{R})$ with $s > 3/2$ by Li and Olver [24] and Rodríguez-Blanco [27]. The global existence of the weak solution in the energy space $H^1(\mathbb{R})$ without any sign conditions on the initial value, and the uniqueness of this weak solution being obtained under some restrictions on the solution were proved by Constantin and Escher [7] and Xin and Zhang [32,33]. Our aim in this paper is to consider the well-posedness of Cauchy problem (1.1)–(1.2).

First, we consider the regularized equation of (1.1)–(1.2) as follows:

$$\partial_t u - \partial_x^2 \partial_t u - \varepsilon\left(1 - \partial_x^2\right)\partial_x^3 u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u = \Phi \frac{\partial^2 B}{\partial t \partial x}, \tag{1.3}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R},\ t > 0, \tag{1.4}$$

where $0 < \varepsilon < 1/4$.

Acting (1.3) with the Bessel potential $(1 - \partial_x^2)^{-1}$ we obtain the following form:

$$\partial_t u - \varepsilon \partial_x^3 u + \frac{1}{2}\partial_x u^2 + \left(1 - \partial_x^2\right)^{-1} \partial_x \left[u^2 + \frac{1}{2}(\partial_x u)^2\right] = \left(1 - \partial_x^2\right)^{-1} \Phi \frac{\partial^2 B}{\partial t \partial x}, \tag{1.5}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R},\ t > 0. \tag{1.6}$$

For fixed ε, the well-posedness of (1.5)–(1.6) will be studied in Bourgain spaces $X_{s,b}$ (see below for a precise definition of $X_{s,b}$) in the paper by modifying techniques developed by [1,22]. When $\varepsilon = -1$, the Cauchy problem of (1.5)–(1.6) without noise in the Sobolev spaces has been investigated by a few authors (see, e.g. [18–20,25,30,34]). For the periodic case, local well-posedness in $H^s(\mathbf{T})$ for $s > 1/2$, for $s = 1/2$ under the restriction of small initial data and $\mathbf{T}$ being the one-dimensional torus, was proved by Himonas and Misiolek [18,19], a similar result was obtained by Byers [5]. For the non-periodic case, Himonas and Misiolek [20] proved the local well-posedness in $H^s(\mathbb{R})$ for $s > 1/2$. The result was improved to be the local well-posedness in $H^s(\mathbb{R})$ for $s > 1/4$ by Liu and Jin [25] by the bilinear estimate method initiated by Bourgain [1]. For both periodic and non-periodic cases, global well-posedness for (1.5)–(1.6) without noise in H^1 follows from the local results and the H^1 energy conservation law. Recently, the global well-posedness in $H^s(\mathbb{R})$ for $s > (5\sqrt{7} - 10)/4$ was proved by Wang and Cui [30] and in $H^s(\mathbb{R})$ for $s > (6\sqrt{10} - 17)/4$ by Yang and Li [34] using the I-method, which is initially developed in [8–10]. However, all of the aforementioned works for Eqs. (1.5)–(1.6) without noise in Bourgain spaces $X_{s,b}$ with $b > 1/2$. Roughly speaking, the index b represents the smoothness in time. However, the stochastic system (1.5)–(1.6) has the same time regularity as Brownian motion with $b < 1/2$. When trying to apply Bourgain's method, we have to encounter spaces $X_{s,b}$ with $b < 1/2$, hence we will have to prove bilinear estimates for the term $\partial_x u^2 + (1 - \partial_x^2)^{-1}\partial_x[u^2 + \frac{1}{2}(\partial_x u)^2]$ in these spaces. In order to get well-posedness of (1.1)–(1.2) from (1.5)–(1.6), we face another difficulty to obtain the regularity estimates. It will be solved by some ideas for the deterministic equation from [24] and some analytic tools from [15].

This paper is organized as follows: in Section 2, we will give some necessary notations and the preliminary estimates; in Section 3, the well-posedness of regularized equation is obtained; in Section 4, regularity estimates of solutions of the regularized equation are proved and a local existence and uniqueness result in the Sobolev space H^s with $s > 3/2$ of stochastic Camassa–Holm equation is obtained; Section 5 is devoted to the proof of blow-up solutions.

2. Preliminary estimates

In this section, the estimate of stochastic integral and the bilinear estimates are obtained. Firstly, some notations are given.

Let us write the Itô form of (1.5)–(1.6) as follows:

$$du + \left(-\varepsilon\partial_x^3 u + \frac{1}{2}\partial_x u^2 + \left(1-\partial_x^2\right)^{-1}\partial_x\left[u^2 + \frac{1}{2}(\partial_x u)^2\right]\right)dt = \left(1-\partial_x^2\right)^{-1}\Phi\, dW, \tag{2.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R},\ t > 0, \tag{2.2}$$

where $W(t,x,\omega) = \frac{\partial B}{\partial x} = \sum_{k=0}^{\infty}\beta_k(t,\omega)e_k(x)$, $(\beta_k)_{k\in\mathbb{N}}$ denotes a sequence of independent Brownian motions in a fixed complete probability space $(\Omega,\mathcal{F},\mathbf{P})$ adapted to a filtration $(\mathcal{F}_t)_{t\geqslant 0}$ and $(e_k)_{k\in\mathbb{N}}$ is an orthogonal basis of $L^2(\mathbb{R})$.

Set $w(x,t) = \frac{1}{2}\partial_x u^2 + (1-\partial_x^2)^{-1}\partial_x[u^2 + \frac{1}{2}(\partial_x u)^2]$, then the mild form of (2.1)–(2.2) is

$$u(t) = U(t)u_0(x) - \int_0^t U(t-s)w(x,s)\,ds + \int_0^t U(t-s)\left(1-\partial_x^2\right)^{-1}\Phi\, dW(s), \tag{2.3}$$

where $U(t)u_0(x) = \int_{\mathbb{R}}\hat{\varphi}(\xi)e^{i(\xi x - \phi(\xi)t)}\,d\xi$, $\phi(\xi) = \varepsilon\xi^3$.

Now, we give the definition of Bourgain spaces.

Definition 2.1. For $s, b \in \mathbb{R}$, we define the space $X_{s,b}$ to be the completion of the Schwartz function space on $\mathbb{R}^2$ with respect to the norm

$$\left\|u(t)\right\|_{X_{s,b}} = \left\|\left(1+\left|\tau+\varepsilon\xi^3\right|\right)^b\left(1+|\xi|\right)^s\hat{u}(\xi,\tau)\right\|_{L_\xi^2 L_\tau^2}.$$

For $T > 0$, $X_{s,b}^T$ is defined by the space restricted to $[0,T]$ of functions in $X_{s,b}$ with norm

$$\|u\|_{X_{s,b}^T} = \inf\left\{\|\tilde{u}\|_{X_{s,b}},\ \tilde{u}\in X_{s,b},\ u = \tilde{u}|_{[0,T]}\right\}.$$

Next, some useful notations for multilinear expressions from [29] are given. Let Z be any Abelian additive group with an invariant measure $d\xi$. For any integer $k \geqslant 2$, we denote by $\Gamma_k(Z)$ the "hyperplane"

$$\Gamma_k(Z) = \left\{(\xi_1,\ldots,\xi_k)\in Z^k\colon\ \xi_1+\cdots+\xi_k = 0\right\},$$

and define a $[k;Z]$-multiplier to be any function $m : \Gamma_k(Z) \to \mathbb{C}$.

If m is a $[k; Z]$-multiplier, we define $\|m\|_{[k;Z]}$ to be the best constant, such that the inequality

$$\left|\int_{\Gamma_k(Z)} m(\xi)\prod_{j=1}^{k} f_j(\xi_j)\right| \leqslant \|m\|_{[k;Z]}\prod_{j=1}^{k}\|f_j\|_{L_2(Z)}$$

holds for all test functions f defined on Z. In this paper $Z = \mathbb{R}\times\mathbb{R}$.

For $x, y \in \mathbb{R}$, $x \lesssim y$ means that there exists $C > 0$, which may vary from line to line and depend on various parameters, such that $x \leqslant Cy$. Let $\psi \in C_c^\infty(\mathbb{R})$ be a decreasing function with $\psi \equiv 1$ on $[0,1]$ and $\operatorname{supp}\psi \subseteq [-1,2]$, and $\psi_\delta(t) = \psi(t/\delta)$, $\delta > 0$. Hereafter, C denotes a positive constant, whose value may change from one place to another.

2.1. Estimate on stochastic integral

The following proposition is devoted to the estimate on stochastic integral in (2.1).

Proposition 2.1. *Let $s, b \in \mathbb{R}$ with $0 < b < 1/2$, and assume $\Phi \in L_2^{0,s-2}$, ψ is a radially decreasing function defined above. Then, for given $t > 0$, $\phi = \int_0^t U(t-s)(1-\partial_x^2)^{-1}\Phi\, dW(s)$ satisfies $\psi\phi \in L^2(\Omega; X_{s,b})$ and*

$$\mathbb{E}\left(\|\psi\phi\|_{X_{s,b}}^2\right) \leqslant C\|\Phi\|_{L_2^{0,s-2}}^2, \tag{2.4}$$

where C is a constant depending only on b, $\|\psi\|_{H^b}$, $\||t|^{1/2}\psi\|_{L^2}$, $\||t|^{1/2}\psi\|_{L^\infty}$.

Proof. The method used in [12] can be applied here with little modification, so we omit it. □

2.2. Bilinear estimates

In this subsection, we first give some lemmas that are useful to prove the bilinear estimates. For convenience, in what follows, we denote

$$\hat{F}_\rho = \frac{\hat{f}(\xi,\tau)}{(1+|\tau+\varepsilon\xi^3|)^\rho}, \qquad \hat{G}_\rho = \frac{\hat{g}(\xi,\tau)}{(1+|\tau+\varepsilon\xi^3|)^\rho}, \qquad \hat{H}_\rho = \frac{\hat{h}(\xi,\tau)}{(1+|\tau+\varepsilon\xi^3|)^\rho}.$$

Lemma 2.1. *(See [17,21,29].) Let $\rho > 3/8$. Then*

$$\left\|D_x^{\frac{1}{8}}F_\rho\right\|_{L_t^4 L_x^4} \lesssim \|f\|_{L_x^2 L_t^2}. \tag{2.5}$$

Let $\rho > \frac{1}{2}\frac{3(q-2)}{2q}$. Then

$$\|F_\rho\|_{L_t^q L_x^q} \lesssim \|f\|_{L_x^2 L_t^2}. \tag{2.6}$$

Lemma 2.2. *(See [12,31].) If $\frac{1}{2} < \ell < 1$, $\jmath > 3/4$, then*

$$\int_{\mathbb{R}} \frac{dx}{(1+|x-\alpha|)^\ell(1+|x-\beta|)^\ell} \lesssim \frac{1}{(1+|\alpha-\beta|)^{2\ell-1}}, \tag{2.7}$$

$$\int_{\mathbb{R}} \frac{dx}{(1+|x|)^\jmath\sqrt{\alpha-x}} \lesssim \frac{1}{(1+|\alpha|)^{1/4}}. \tag{2.8}$$

Lemma 2.3. *(See [29].) If m and M are $[k; Z]$-multipliers and satisfy $|m(\xi)| \leqslant |M(\xi)|$ for all $\xi \in \Gamma_k(Z)$, then* $\|m\|_{[k;Z]} \leqslant \|M\|_{[k;Z]}$.

Now, the bilinear estimates for $w(x,t)$ in Bourgain spaces can be obtained in the following Propositions 2.2–2.4.

Proposition 2.2. *Let $1/2 < b < 5/8$ and $7/16 < b' < 1/2$. Then, for $s \geqslant -1/8$, we have*

$$\left\| \partial_x(uv) \right\|_{X_{s,b-1}} \leqslant C \|u\|_{X_{s,b'}} \|v\|_{X_{s,b'}}. \tag{2.9}$$

Proof. For convenience, we define

$$\hat{f}(\xi,\tau) = \langle\xi\rangle^s \langle\sigma\rangle^{b'} \hat{u}(\xi,\tau), \qquad \hat{g}(\xi,\tau) = \langle\xi\rangle^s \langle\sigma\rangle^{b'} \hat{v}(\xi,\tau), \qquad k(\tau,\xi) = \frac{\langle\xi_1\rangle^{-s} \langle\xi_2\rangle^{-s}}{\langle\sigma_1\rangle^{b'} \langle\sigma_2\rangle^{b'}},$$

where $\langle\xi\rangle = (1+|\xi|)$, $\sigma = \sigma(\tau,\xi) = \tau + \varepsilon\xi^3$, $\sigma_1 = \sigma(\tau_1,\xi_1)$, $\sigma_2 = \sigma(\tau_2,\xi_2)$, $\xi_2 = \xi - \xi_1$, $\tau_2 = \tau - \tau_1$.

Then (2.9) is equivalent to the estimate

$$\left\| \langle\xi\rangle^s \langle\sigma\rangle^{b-1} |\xi| \int_{\mathbb{R}} \int_{\mathbb{R}} k(\tau,\xi) \hat{f}(\xi_1,\tau_1) \hat{g}(\xi_2,\tau_2)\, d\xi_1\, d\tau_1 \right\|_{L^2_\xi L^2_\tau} \lesssim \|f\|_{L^2} \|g\|_{L^2}. \tag{2.10}$$

Hence by the self-duality of L^2, it is easy to see that (2.10) is equivalent to

$$\left| \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} k_1(\tau,\xi,\tau_1,\xi_1) \hat{f}(\xi_1,\tau_1) \hat{g}(\xi_2,\tau_2) \hat{h}(\xi,\tau)\, d\xi_1\, d\tau_1\, d\xi\, d\tau \right| \lesssim \|f\|_{L^2} \|g\|_{L^2} \|h\|_{L^2},$$

where

$$k_1(\tau,\xi,\tau_1,\xi_1) = \frac{|\xi| \langle\xi\rangle^s \langle\xi_1\rangle^{-s} \langle\xi_2\rangle^{-s}}{\langle\sigma\rangle^{1-b} \langle\sigma_1\rangle^{b'} \langle\sigma_2\rangle^{b'}}.$$

By symmetry it suffices to estimate the integral in the domain $|\xi_1| \leqslant |\xi_2|$.

Case 1. $|\xi_1| \leqslant 1$ or $|\xi_2| \leqslant 1$.

In this case, it can easily be obtained that

$$\frac{\langle\xi\rangle^s |\xi|}{\langle\xi_2\rangle^s \langle\xi_1\rangle^s} \lesssim |\xi|.$$

Using the Cauchy–Schwarz inequality, and Fubini's theorem it follows that

$$\begin{aligned}
&\left\| \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\langle\sigma\rangle^{b-1} |\xi|}{\langle\sigma_2\rangle^{b'} \langle\sigma_1\rangle^{b'}} \hat{f}(\xi_1,\tau_1) \hat{g}(\xi_2,\tau_2)\, d\xi_1\, d\tau_1 \right\|_{L^2_\xi L^2_\tau} \\
&\quad \lesssim \left\| \langle\sigma\rangle^{b-1} |\xi| \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{\langle\sigma_1\rangle^{2b'} \langle\sigma_2\rangle^{2b'}}\, d\xi_1\, d\tau_1 \right)^{1/2} \right\|_{L^\infty_\xi L^\infty_\tau} \\
&\quad\quad \times \left\| \left(\int_{\mathbb{R}} \int_{\mathbb{R}} f^2(\xi_1,\tau_1) g^2(\xi_2,\tau_2)\, d\xi_1\, d\tau_1 \right)^{1/2} \right\|_{L^2_\xi L^2_\tau}.
\end{aligned}$$

Applying inequality (2.7), we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\langle\sigma\rangle^{2(b-1)}|\xi|^2}{\langle\sigma_2\rangle^{2b'}\langle\sigma_1\rangle^{2b'}}\,d\xi_1\,d\tau_1 \lesssim \langle\sigma\rangle^{2(b-1)}|\xi|^2\int_{\mathbb{R}} \frac{d\xi_1}{(1+|\tau+\varepsilon(\xi^3-3\xi^2\xi_1+3\xi\xi_1^2)|)^{4b'-1}}.$$

To integrate with respect to ξ_1 we change variables

$$\mu = \tau+\varepsilon\big(\xi^3-3\xi^2\xi_1+3\xi\xi_1^2\big),$$

so we have

$$d\mu = 3\varepsilon\xi(2\xi_1-\xi)\,d\xi_1, \quad \text{and} \quad \xi_1 = \frac{\xi}{2}\pm\frac{\sqrt{-4\tau-\varepsilon\xi^3+4\mu}}{2\sqrt{3}\sqrt{\varepsilon\xi}}.$$

Then applying (2.8) we yield, for $b' > 7/16$

$$\begin{aligned}\int_{\mathbb{R}} \frac{d\xi_1}{(1+|\tau+\varepsilon(\xi^3-3\xi^2\xi_1+3\xi\xi_1^2)|)^{4b'-1}} &= \frac{2}{\sqrt{3\varepsilon\xi}}\int_{\mathbb{R}} \frac{d\mu}{(1+\mu)^{4b'-1}\sqrt{-(\varepsilon\xi^3+4\tau)+4\mu}} \\ &\lesssim \frac{1}{\sqrt{\xi}(1+|\tau+\frac{\varepsilon\xi^3}{4}|)^{1/4}}. \qquad (2.11)\end{aligned}$$

Combining this with the previous estimate yields, for $b < 7/8$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\langle\sigma\rangle^{2(b-1)}|\xi|^2}{\langle\sigma_2\rangle^{2b'}\langle\sigma_1\rangle^{2b'}}\,d\xi_1\,d\tau_1 \lesssim \frac{(1+|\tau+\xi^3|)^{2(b-1)}|\xi|^{3/2}}{(1+|\tau+\frac{\varepsilon\xi^3}{3}|)^{1/4}} \lesssim 1.$$

Case 2. $|\xi_2| \geqslant |\xi_1| \geqslant 1$, $|\xi| \leqslant 2$.

We denote by J_1 the left integral of (2.11) restricted on this region.

$$k_1(\tau,\xi,\tau_1,\xi_1) \lesssim \frac{\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}}{\langle\sigma\rangle^{1-b}\langle\sigma_1\rangle^{b'}\langle\sigma_2\rangle^{b'}}.$$

Hence, for $s \geqslant -1/8$, $b' > -3/8$, $b < 1$, using Hölder's inequality and Lemma 2.1, we get

$$\begin{aligned}J_1 &\lesssim \left|\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\hat{f}(\xi_1,\tau_1)\hat{g}(\xi_2,\tau_2)\hat{h}(\xi,\tau)\langle\xi_1\rangle^{-s}\langle\xi_2\rangle^{-s}}{\langle\sigma\rangle^{1-b}\langle\sigma_1\rangle^{b'}\langle\sigma_2\rangle^{b'}}\,d\xi_1\,d\tau_1\,d\xi\,d\tau\right| \\ &\lesssim \big\|D_x^{1/8}F_{b'}\big\|_{L_x^4L_t^4}\big\|D_x^{1/8}G_{b'}\big\|_{L_x^4L_t^4}\|H_{1-b}\|_{L_x^2L_t^2} \\ &\lesssim \|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}.\end{aligned}$$

Case 3. $|\xi_2| \geqslant |\xi_1| \geqslant 1$, $|\xi| \geqslant 2$.

We denote by J_2 the left integral of (2.11) restricted on this region. Since

$$\sigma - \sigma_1 - \sigma_2 = 3\varepsilon\xi\xi_1\xi_2,$$

one of the following cases occurs:

(a) $|\sigma| \gtrsim |\xi||\xi_1||\xi_2|$;
(b) $|\sigma_1| \gtrsim |\xi||\xi_1||\xi_2|$;
(c) $|\sigma_2| \gtrsim |\xi||\xi_1||\xi_2|$.

If (a) holds, we have

$$k_1(\tau, \xi, \tau_1, \xi_1) \lesssim \frac{|\xi|^{s+b}\langle\xi_1\rangle^{b-s-1}\langle\xi_2\rangle^{b-s-1}}{\langle\sigma_1\rangle^{b'}\langle\sigma_2\rangle^{b'}},$$

by Hölder's inequality and Lemma 2.1, for $b' < 3/8$, $s + b \leqslant 0$ and $b - s - 1 \leqslant 1/8$, we obtain

$$\begin{aligned} J_{21} &\lesssim \left| \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\hat{f}(\xi_1, \tau_1)\hat{g}(\xi_2, \tau_2)\hat{h}(\xi, \tau)|\xi|^{s+b}\langle\xi_1\rangle^{b-s-1}\langle\xi_2\rangle^{b-s-1}}{\langle\sigma_1\rangle^{b'}\langle\sigma_2\rangle^{b'}}\, d\xi_1\, d\tau_1\, d\xi\, d\tau \right| \\ &\lesssim \left\| D_x^{1/8} F_{b'} \right\|_{L_x^4 L_t^4} \left\| D_x^{1/8} G_{b'} \right\|_{L_x^4 L_t^4} \|H_0\|_{L_x^2 L_t^2} \\ &\lesssim \|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}. \end{aligned}$$

This means that if $b - s - 1 \leqslant 1/8$, $s \leqslant -b$, we have

$$\left\| k_1(\tau, \xi, \tau_1, \xi_1) \right\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim 1.$$

However, if $s \geqslant -1/8$, then by Lemma 2.3, we have

$$\left\| k_1(\tau, \xi, \tau_1, \xi_1) \right\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim 1.$$

In fact, if $s_1 \leqslant s_2$, then

$$\frac{|\xi|\langle\xi\rangle^{s_2}\langle\xi_1\rangle^{-s_2}\langle\xi_2\rangle^{-s_2}}{\langle\sigma\rangle^{1-b}\langle\sigma_1\rangle^{b'}\langle\sigma_2\rangle^{b'}} \leqslant \frac{|\xi|\langle\xi\rangle^{s_1}\langle\xi_1\rangle^{-s_1}\langle\xi_2\rangle^{-s_1}}{\langle\sigma\rangle^{1-b}\langle\sigma_1\rangle^{b'}\langle\sigma_2\rangle^{b'}}.$$

If (b) holds, using Hölder's inequality and Lemma 2.1, we obtain

$$k_1(\tau, \xi, \tau_1, \xi_1) \lesssim \frac{|\xi|^{1+s-b'}\langle\xi_2\rangle^{2(-s-b')}}{\langle\sigma\rangle^{1-b}\langle\sigma_2\rangle^{b'}},$$

and for $b' > 3/8$, $b < 5/8$, $1 + s - b' \leqslant 0$ and $-b' - s \leqslant 1/16$

$$\begin{aligned} J_{22} &\lesssim \left| \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\hat{f}(\xi_1, \tau_1)\hat{g}(\xi_2, \tau_2)\hat{h}(\xi, \tau)|\xi|^{1+s-b'}\langle\xi_2\rangle^{2(-s-b')}}{\langle\sigma\rangle^{1-b}\langle\sigma_2\rangle^{b'}}\, d\xi_1\, d\tau_1\, d\xi\, d\tau \right| \\ &\lesssim \|F_{1-b}\|_{L_x^4 L_t^4} \left\| D_x^{1/8} G_{b'} \right\|_{L_x^4 L_t^4} \|H_0\|_{L_x^2 L_t^2} \\ &\lesssim \|f\|_{L^2}\|g\|_{L^2}\|h\|_{L^2}. \end{aligned}$$

Similar to (a), if $s \geqslant -1/8$, by Lemma 2.3, we have

$$\left\| k_1(\tau, \xi, \tau_1, \xi_1) \right\|_{[3;\mathbb{R}\times\mathbb{R}]} \lesssim 1.$$

The argument for (c) is similar to (b). This completes the proof of Proposition 2.2. □

As the similar proof of Proposition 2.2, we can get the following Propositions 2.3–2.4.

Proposition 2.3. *Let $1/2 < b < 5/8$ and $7/16 < b' < 1/2$. Then, for $s \geqslant -1/8$, we have*

$$\left\| \left(1 - \partial_x^2\right)^{-1} \partial_x(uv) \right\|_{X_{s,b-1}} \leqslant C \|u\|_{X_{s,b'}} \|v\|_{X_{s,b'}}. \tag{2.12}$$

Proposition 2.4. *Let $1/2 < b < 5/8$ and $7/16 < b' < 1/2$. Then, for $s \geqslant 7/8$, we get*

$$\left\| \left(1 - \partial_x^2\right)^{-1} \partial_x(u_x v_x) \right\|_{X_{s,b-1}} \leqslant C \|u\|_{X_{s,b'}} \|v\|_{X_{s,b'}}. \tag{2.13}$$

3. Well-posedness of regularized equation

In this section, the existence and uniqueness of solutions for the regularized initial value problem (1.3)–(1.4) is proved by a fixed point argument in the space $X_{s,b}$. First we define a solution of (1.3)–(1.4) and a solution of (1.1)–(1.2).

Definition 3.1. Let $s \geqslant 7/8$. An $H^s(\mathbb{R})$-value progressively measurable stochastic process u is a solution of (1.3)–(1.4) if $u \in C([0,T]; H^s(\mathbb{R}))$ for a.e. $\omega \in \Omega$, and

$$\begin{aligned}\int_{\mathbb{R}} u\varphi\,dx = {} & \int_0^t \int_{\mathbb{R}} u_0 \varphi\,dx\,ds + \varepsilon \int_0^t \int_{\mathbb{R}} u \partial_x^3 \varphi\,dx\,ds - \frac{1}{2} \int_0^t \int_{\mathbb{R}} u^2 \partial_x \varphi\,dx\,ds \\ & - \int_0^t \int_{\mathbb{R}} \left(1 - \partial_x^2\right)^{-1} \left[u^2 + \frac{1}{2} (\partial_x u)^2 \right] \partial_x \varphi\,dx\,ds + \int_0^t \int_{\mathbb{R}} \varphi \left(1 - \partial_x^2\right)^{-1} \Phi\,dW\,dx\end{aligned}$$

holds for all $\varphi \in C_0^\infty(\mathbb{R})$, for all $t \in [0,T]$ and for a.e. $\omega \in \Omega$.

Definition 3.2. Let $s > 3/2$. An $H^s(\mathbb{R})$-value progressively measurable stochastic process u is a solution of (1.1)–(1.2) if $u \in C([0,T]; H^s(\mathbb{R}))$ for a.e. $\omega \in \Omega$, and

$$\begin{aligned}\int_{\mathbb{R}} u\varphi\,dx = {} & \int_0^t \int_{\mathbb{R}} u_0 \varphi\,dx\,ds - \frac{1}{2} \int_0^t \int_{\mathbb{R}} u^2 \partial_x \varphi\,dx\,ds \\ & - \int_0^t \int_{\mathbb{R}} \left(1 - \partial_x^2\right)^{-1} \left[u^2 + \frac{1}{2} (\partial_x u)^2 \right] \partial_x \varphi\,dx\,ds + \int_0^t \int_{\mathbb{R}} \varphi \left(1 - \partial_x^2\right)^{-1} \Phi\,dW\,dx\end{aligned}$$

holds for all $\varphi \in C_0^\infty(\mathbb{R})$, for all $t \in [0,T]$ and for a.e. $\omega \in \Omega$.

The following lemma is needed for applying fixed point argument.

Lemma 3.1. *(See [12,21].) Let* $0 < a, b < 1/2 < c < 1$, $s \in \mathbb{R}$, $u_0 \in H^s(\mathbb{R})$, $f \in X^T_{s,c}$. *Then for any* $T \in [0,1]$, $t \in [0,T]$, *we have*

$$\left\| U(t)u_0 \right\|_{X^T_{s,c}} \leqslant C\|u_0\|_{H^s}, \tag{3.1}$$

$$\left\| \int_0^t U(t-s)f(s)\,ds \right\|_{X^T_{s,b}} \leqslant CT^{1-a-b}\|f\|_{X_{s,-a}}. \tag{3.2}$$

The main result of this section is as follows:

Theorem 3.1. *Assume that for some s with* $s \geqslant 7/8$, $\Phi \in L_2^{0,s-2}$, *let* $u_0(\omega, x)$ *be such that* $u_0 \in H^s(\mathbb{R})$ *for a.e.* $\omega \in \Omega$ *and* u_0 *being* $\mathcal{F}_0$*-measurable. Then, for a.e.* $\omega \in \Omega$, *there is a* $T_\omega > 0$ *and a unique solution* $u(t)$ *of* (1.3)–(1.4) *on* $[0, T_\omega]$ *which satisfies*

$$u \in C\big([0,T_\omega]; H^s(\mathbb{R})\big) \cap X^{T_\omega}_{s,b}.$$

Proof. From Proposition 2.1, we get

$$\|\phi\|_{X^T_{s,b}} \leqslant C\|\psi\phi\|_{X_{s,b}}, \tag{3.3}$$

for $\phi = \int_0^t U(t-s)(1-\partial_x^2)^{-1}\Phi\,dW(s)$, ψ given before and $T \in [0,1]$ for almost each $\omega \in \Omega$. We fix $\omega \in \Omega$ such that (3.3) and $u_0(\omega,\cdot) \in H^s(\mathbb{R})$ hold. Set

$$z(t) = U(t)u_0, \qquad v(t) = u(t) - \phi(t) - z(t).$$

Then Eqs. (2.1)–(2.2) can be rewritten in terms of

$$v(t) = -\int_0^t U(t-s)\frac{1}{2}\partial_x(v+\phi+z)^2 + \big(1-\partial_x^2\big)^{-1}\partial_x\left[(v+\phi+z)^2 + \frac{1}{2}\big(\partial_x(v+\phi+z)^2\big)\right]ds.$$

Let us introduce the complete metric space

$$B^T_R = \big\{ v \in X^T_{s,b},\ \|v\|_{X^T_{s,b}} \leqslant R \big\}, \tag{3.4}$$

with $R = \|\phi\|_{X^T_{s,b}} + \|u_0\|_{H^s}$. We set

$$\begin{aligned}\Gamma v(t) = &-\int_0^t U(t-s)\frac{1}{2}\partial_x(v+\phi+z)^2 \\ &+ \big(1-\partial_x^2\big)^{-1}\partial_x\left[(v+\phi+z)^2 + \frac{1}{2}\big(\partial_x(v+\phi+z)^2\big)\right]ds.\end{aligned} \tag{3.5}$$

We will show that Γ is a contraction mapping in B^T_R, for $R > 0$ sufficiently large, provided that T is chosen sufficiently small. With this aim in view, let $v, v_1, v_2 \in B^T_R$ be adapted processes. Noticing Propositions 2.1–2.4, and Lemma 3.1, we easily get

$$\|\Gamma v\|_{X^T_{s,b}} \leqslant CT^{1-a-b}\big(R^2 + \|\phi\|^2_{X^T_{s,b}} + \|u_0\|^2_{H^s}\big),$$

$$\|\Gamma v_1 - \Gamma v_2\|_{X^T_{s,b}} \leqslant CT^{1-a-b}\big(R + \|\phi\|_{X^T_{s,b}} + \|u_0\|_{H^s}\big)\|v_1 - v_2\|_{X^T_{s,b}}.$$

Define the stopping time T by

$$T_\omega = \inf\big\{t > 0,\ 2Ct^{1-a-b}R \leqslant 1/2\big\}.$$

Then Γ maps $B^{T_\omega}_R$ in $X^{T_\omega}_{s,b}$ into itself, and

$$\|\Gamma v_1 - \Gamma v_2\|_{X^{T_\omega}_{s,b}} \leqslant \frac{1}{2}\|v_1 - v_2\|_{X^{T_\omega}_{s,b}}.$$

Thus the contraction mapping principle implies that there exists a unique solution u in $X^{T_\omega}_{s,b}$ on $[0, T_\omega]$ to problem (3.5). It remains to show that the solution $u = z + v + \phi \in X^{T_\omega}_{s,c} + X^{T_\omega}_{s,b}$ in $C([0, T_\omega], H^s(\mathbb{R}))$ (note that here $b < 1/2$, $c > 1/2$). Since $c > 1/2$, we have $z \in C([0, T_\omega], H^s(\mathbb{R}))$ by the Sobolev imbedding theorem in time. Since $\Phi \in L_2^{0,s}$ and $U(\cdot)$ is a unitary group in $H^s(\mathbb{R})$, we have that ϕ has a continuous modification with values in $H^s(\mathbb{R})$ similar to that of Theorem 6.10 in Ref. [15].

Let $\tilde{u}$ be any prolongation of u in $X_{s,c} + X_{s,b}$, by Propositions 2.2–2.4, $\frac{1}{2}\partial_x\tilde{u}^2 + (1-\partial_x^2)^{-1}\partial_x[\tilde{u}^2 + \frac{1}{2}(\partial_x\tilde{u})^2] \in X_{s,-a}$ with $-\frac{1}{2} < a < 0$. It follows that (see Ref. [21])

$$\begin{aligned}&\left\|\varphi_T\int_0^t U(t-s)\left(\frac{1}{2}\partial_x\tilde{u}^2 + \big(1-\partial_x^2\big)^{-1}\partial_x\left[\tilde{u}^2 + \frac{1}{2}(\partial_x\tilde{u})^2\right]\right)ds\right\|_{X_{s,1-a}}\\ &\quad\lesssim \big\|\partial_x\big(\tilde{u}^2\big)\big\|_{X_{s,-a}} + \left\|\big(1-\partial_x^2\big)^{-1}\partial_x\left[\tilde{u}^2 + \frac{1}{2}(\partial_x\tilde{u})^2\right]\right\|_{X_{s,-a}}.\end{aligned} \tag{3.6}$$

Since $1 - a > 1/2$, $\tilde{u} \in X_{s,1-a} \subset C([0, T_\omega], H^s(\mathbb{R}))$, where φ_T is a cut-off function defined by $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi = 1$ on $[0, 1]$, and $\varphi = 0$ on $t \leqslant -1$, $t \geqslant 2$. Denote $\varphi_\delta(\cdot) = \varphi(\delta^{-1}(\cdot))$ for some $\delta \in \mathbb{R}$. This ends the proof of this Theorem 3.1. □

4. Regularity estimates of solutions

In this section, the regularity estimates of solutions to the regularized initial value u_0 and the regularized equations as (4.15)–(4.16) are established. Then, a local well-posedness result in the Sobolev space H^s with $s > 3/2$ of stochastic Camassa–Holm equation (1.1)–(1.2) is proved by the regularity estimates. The following Lemmas 4.1–4.3 are important to obtain the desired estimates.

Lemma 4.1. *(See [23].) If $r > 0$, then $H^r \cap L^\infty$ is an algebra. Moreover*

$$\|uv\|_r \lesssim \|u\|_{L^\infty}\|v\|_r + \|u\|_r\|v\|_{L^\infty}. \tag{4.1}$$

Lemma 4.2. *(See [23].) Let $r > 0$, if $u \in H^r \cap W^{1,\infty}$ and $v \in H^{r-1} \cap L^\infty$, then*

$$\big\|\big[\Lambda^r, u\big]v\big\|_{L^2} \lesssim \|\partial_x u\|_{L^\infty}\big\|\Lambda^{r-1}v\big\|_{L^2} + \|\Lambda^r u\|_{L^2}\|v\|_{L^\infty}, \tag{4.2}$$

where $[\Lambda^r, u]v = \Lambda^r(uv) - u\Lambda^r v$, $\Lambda = (1-\partial_x^2)^{1/2}$.

Lemma 4.3. *(See [24].) Given $q \geqslant 0$, let $u = u(x) \in H^q$ be any function such that $\|u_x\|_{L^\infty} < \infty$. Then there is a constant c_q depending only on q such that the following inequalities hold,*

$$\left| \int_{\mathbb{R}} \Lambda^q u \Lambda^q \left(u^2\right) dx \right| \leqslant c_q \|u\|_{L^\infty} \|u_x\|_{H^q}^2.$$

If u and f are functions in $H^{q+1} \cap \{\|u_x\|_{L^\infty} < \infty\}$, then

$$\left| \int_{\mathbb{R}} \Lambda^q u \Lambda^q (uf)_x \, dx \right| \leqslant \begin{cases} c_q \|f\|_{H^{q+1}} \|u\|_{H^q}^2, & q \in (1/2, 1], \\ c_q (\|f\|_{H^{q+1}} \|u\|_{H^q} \|u\|_{L^\infty} + \|f_x\|_{L^\infty} \|u\|_{H^q}^2 & \\ \quad + \|f\|_{H^q} \|u\|_{H^q} \|u_x\|_{L^\infty}), & q \in (0, \infty). \end{cases}$$

Now, some priori estimates of a solution of the problem (1.3)–(1.4) are given.

Lemma 4.4. *Let $s \geqslant 5$ and the function $u(t,x)$ is a solution of the problem* (1.3)–(1.4) *with the initial data $u_0(x) \in H^s$, $\Phi \in L_0^{2,s}$. Then, for $q \in (0, s-1]$, we have*

$$\mathbb{E} \sup_{t \in [0,T]} \|u\|_{H^1}^2 \lesssim \mathbb{E}\|u_0\|_{H^1}^2 + T\|\Phi\|_{L_2^{0,0}}^2, \tag{4.3}$$

$$\mathbb{E} \sup_{t \in [0,T]} \|u\|_{H^{q+1}}^2 \lesssim \mathbb{E}\|u_0\|_{H^{q+1}}^2 + \mathbb{E} \int_0^T \|u_x\|_{L^\infty} \left(\|u\|_{H^q}^2 + \|u\|_{H^{q+1}}^2\right) ds + T\|\Phi\|_{L_2^{0,q}}^2. \tag{4.4}$$

Proof. By regularizing the initial value u_0 and the operator Φ as in (4.15)–(4.16) (or the regularization argument in [13,14]), then using the Itô formula (see, for example [15]) to $\|u\|_{L^2}^2$ and some computations, we have

$$\begin{aligned} \|u\|_{L^2}^2 = {} & \|u_0\|_{L^2}^2 + 2\int_0^t \int_{\mathbb{R}} u \partial_s \partial_x^2 u \, dx \, ds + 2\varepsilon \int_0^t \int_{\mathbb{R}} u \partial_x^3 u \, dx \, ds - 2\varepsilon \int_0^t \int_{\mathbb{R}} u \partial_x^5 u \, dx \, ds \\ & - 6\int_0^t \int_{\mathbb{R}} u^2 \partial_x^2 u \, dx \, ds + 4\int_0^t \int_{\mathbb{R}} u \partial_x u \partial_x^2 u \, dx \, ds + 2\int_0^t \int_{\mathbb{R}} u^2 \partial_x^3 u \, dx \, ds \\ & + 2\int_0^t \int_{\mathbb{R}} u\Phi \, dW \, dx + \int_0^t \|\Phi\|_{L_2^{0,0}}^2 \, ds. \end{aligned} \tag{4.5}$$

Using integration by parts, we have

$$\|u\|_{H^1}^2 = \|u_0\|_{H^1}^2 + 2\int_0^t \int_{\mathbb{R}} u\Phi \, dW \, dx + \int_0^t \|\Phi\|_{L_2^{0,0}}^2 \, ds. \tag{4.6}$$

By the Burkholder–Davis–Gundy (B–D–G) inequality (see, e.g., [15]), we have

$$\mathbb{E}\sup_{t\in[0,T]}\left|\int_0^t\int_{\mathbb{R}} u\Phi\,dW\,dx\right| \leqslant 3\mathbb{E}\left\{\int_0^T\int_{\mathbb{R}}|u\Phi|^2\,dx\,dt\right\}^{1/2}$$
$$\leqslant 3\mathbb{E}\left\{\sup_{t\in[0,T]}\|u\|_{L^2}^2\int_0^T\|\Phi\|_{L_2^{0,0}}^2\,dt\right\}^{1/2}$$
$$\leqslant \frac{1}{4}\mathbb{E}\sup_{t\in[0,T]}\|u\|_{L^2}^2+9T\|\Phi\|_{L_2^{0,0}}^2. \tag{4.7}$$

Taking account of (4.6)–(4.7), we obtain

$$\mathbb{E}\sup_{t\in[0,T]}\|u\|_{H^1}^2\lesssim\mathbb{E}\|u_0\|_{H^1}^2+T\|\Phi\|_{L_2^{0,0}}^2, \tag{4.8}$$

which derives (4.3).

For $q\in(0,s-1]$, applying the Itô formula on $\|u\|_{H^q}^2$, we have the equation

$$\|u\|_{H^q}^2+\|u\|_{H^{q+1}}^2=\|u_0\|_{H^q}^2+\|u_0\|_{H^{q+1}}^2+2\varepsilon\int_0^t\int_{\mathbb{R}}\Lambda^q u\Lambda^q\partial_x^3u\,dx\,ds$$
$$-2\varepsilon\int_0^t\int_{\mathbb{R}}\Lambda^q u\Lambda^q\partial_x^5u\,dx\,ds-6\int_0^t\int_{\mathbb{R}}\Lambda^q u\Lambda^q(u\partial_xu)\,dx\,ds$$
$$-2\int_0^t\int_{\mathbb{R}}\left(\Lambda^{q+1}u\right)\Lambda^{q+1}(u\partial_xu)\,dx\,ds+\int_0^t\int_{\mathbb{R}}\left(\Lambda^q\partial_xu\right)\Lambda^q\left(u_x^2\right)dx\,ds$$
$$+\int_0^t\|\Phi\|_{L_2^{0,q}}^2\,ds+2\int_0^t\int_{\mathbb{R}}\Lambda^q u\Lambda^q\Phi\,dW\,dx, \tag{4.9}$$

where we have used the equality $\int_{\mathbb{R}}2u_xu_{xx}+uu_{xxx}\,dx=\int_{\mathbb{R}}(uu_x)_x-\frac{1}{2}(u_x^2)_x\,dx$.

Using integration by parts, we have

$$\int_{\mathbb{R}}\Lambda^q u\Lambda^q\partial_x^3u\,dx=0,\qquad\int_{\mathbb{R}}\Lambda^q u\Lambda^q\partial_x^5u\,dx=0, \tag{4.10}$$

using integration by parts, the Cauchy–Schwartz inequality, and Lemmas 4.1–4.2, we have

$$\int_{\mathbb{R}}\Lambda^q u\Lambda^q(u\partial_xu)\,dx=\int_{\mathbb{R}}\Lambda^q u\left[\Lambda^q(u\partial_xu)-u\Lambda^q\partial_xu\right]dx+\int_{\mathbb{R}}\left(\Lambda^q u\right)u\Lambda^q\partial_xu\,dx$$
$$\lesssim\|u_x\|_{L^\infty}\|u\|_{H^q}^2, \tag{4.11}$$

$$\int_{\mathbb{R}}\left(\Lambda^{q+1}u\right)\Lambda^{q+1}(u\partial_xu)\,dx\lesssim\|u_x\|_{L^\infty}\|u\|_{H^{q+1}}^2, \tag{4.12}$$

by Lemma 4.3, we have

$$\int_{\mathbb{R}} \big(\Lambda^q \partial_x u\big) \Lambda^q \big(u_x^2\big) \lesssim \|u_x\|_{L^\infty} \|u\|_{H^{q+1}}^2. \tag{4.13}$$

It follows from (4.10)–(4.13) that

$$\begin{aligned} \|u\|_{H^q}^2 + \|u\|_{H^{q+1}}^2 &\lesssim \|u_0\|_{H^{q+1}}^2 + \int_0^t \|u_x\|_{L^\infty} \big(\|u\|_{H^q}^2 + \|u\|_{H^{q+1}}^2\big)\, ds \\ &\quad + \int_0^t \|\Phi\|_{L_2^{0,q}}^2\, ds + 2\int_0^t \int_{\mathbb{R}} \Lambda^q u \Lambda^q \Phi\, dW\, dx. \end{aligned} \tag{4.14}$$

From (4.14), the proof of inequality (4.4) is analogous to (4.3). So we end the proof of Lemma 4.4. □

Define $u_{0\eta} = u_0 * \rho_\eta$, $\Phi_\eta = \Phi * \rho_\eta$, where ρ_η is the Friedrichs mollifier in the space variable $\rho_\eta(x) = \eta^{-\frac{1}{4}} \rho(\eta^{-\frac{1}{4}} x)$ and the Fourier transform $\hat{\rho}$ of ρ satisfies $\hat{\rho} \in C_0^\infty$, $\hat{\rho(\xi)} \geqslant 0$ and $\hat{\rho(\xi)} = 1$ for any $\xi \in (-1, 1)$. Thus we have $u_{0\eta} \in C^\infty$. It follows from Section 3 that for each ε, η satisfying $0 < \varepsilon, \eta < 1/4$, the Cauchy problem

$$\partial_t u + \partial_x^2 \partial_t u - \varepsilon\big(1 - \partial_x^2\big)\partial_x^3 u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u = \Phi_\eta \frac{\partial^2 B}{\partial t \partial x}, \tag{4.15}$$

$$u(x, 0) = u_{0\eta}, \quad x \in \mathbb{R},\ t > 0, \tag{4.16}$$

has a unique solution $u_{\varepsilon,\eta}(x, t) \in L^2(\Omega; C([0, T]; H^\infty(\mathbb{R}))) \cap X_{\infty,b}^T$.

Without loss of generalization, we set $\eta = \varepsilon$ and $u_\varepsilon = u_{\varepsilon,\eta}$ in the following. Next, we will show that u_ε is convergent to a solution of the problem (1.1)–(1.2) in $L^2(\Omega; C([0, T]; H^s(\mathbb{R})))$ with $s > 3/2$. By the following Lemma 4.7, there exists a T_0 independent of ε, such that for any $\varepsilon > 0$, $T > T_0$, (4.15)–(4.16) have solution on the time interval $[0, T_0]$. Hereafter, we still denote T_0 by T.

Lemma 4.5. *Under the above assumptions, the following estimates hold for any ε satisfying $0 < \varepsilon < 1/4$ and $s > 0$,*

$$\mathbb{E}\|u_{0\varepsilon}\|_{H^q}^2 \leqslant c, \quad \text{if } q \leqslant s, \tag{4.17}$$

$$\mathbb{E}\|u_{0\varepsilon}\|_{H^q}^2 \leqslant c\varepsilon^{\frac{s-q}{2}}, \quad \text{if } q > s, \tag{4.18}$$

$$\mathbb{E}\|u_{0\varepsilon} - u_0\|_{H^q}^2 \leqslant c\varepsilon^{\frac{s-q}{2}}, \quad \text{if } q \leqslant s, \tag{4.19}$$

$$\mathbb{E}\|u_{0\varepsilon} - u_0\|_{H^s} = o(1), \tag{4.20}$$

where c is a constant independent of ε.

Proof. The proof is similar to that of Lemma 5 in [2]. □

Similar to Lemma 4.5, we have the following lemma.

Lemma 4.6. *Under the above assumptions, the following estimates for $\Phi_\varepsilon = \Phi * \rho_\varepsilon$ defined above hold for any ε satisfying $0 < \varepsilon < 1/4$ and $s > 0$,*

$$\|\Phi_\varepsilon\|^2_{L_2^{0,q}} \leqslant c, \quad \text{if } q \leqslant s, \tag{4.21}$$

$$\|\Phi_\varepsilon\|^2_{L_2^{0,q}} \leqslant c\varepsilon^{\frac{s-q}{2}}, \quad \text{if } q > s, \tag{4.22}$$

$$\|\Phi_\varepsilon - \Phi\|^2_{L_2^{0,q}} \leqslant c\varepsilon^{\frac{s-q}{2}}, \quad \text{if } q \leqslant s, \tag{4.23}$$

$$\|\Phi_\varepsilon - \Phi\|^2_{L_2^{0,s}} = o(1), \tag{4.24}$$

where c is a constant independent of ε.

Proof. Let $v \in L^2$, using the Fourier transform leads to

$$\rho_\varepsilon(\xi) = \int e^{-ix\xi}\varepsilon^{-\frac{1}{4}}\rho\big(\varepsilon^{-\frac{1}{4}}x\big)\,dx = \int e^{i(\varepsilon^{-\frac{1}{4}}x)(\varepsilon^{\frac{1}{4}}\xi)}\rho\big(\varepsilon^{-\frac{1}{4}}x\big)\,d\big(\varepsilon^{-\frac{1}{4}}x\big) = \hat{\rho}\big(\varepsilon^{\frac{1}{4}}\xi\big).$$

Furthermore, we have

$$\widehat{\Phi_\varepsilon v}(\xi) = \widehat{\rho_\varepsilon * \Phi} v = \hat{\rho_\varepsilon}(\xi)\hat{\Phi} v(\xi) = \hat{\rho}\big(\varepsilon^{\frac{1}{4}}\xi\big)\hat{\Phi} v(\xi),$$

and

$$\begin{aligned}
\|\Phi_\varepsilon v\|^2_{H^q} &= \int_{\mathbb{R}} \big(1+|\xi|^2\big)^q \big|\hat{\rho}\big(\varepsilon^{\frac{1}{4}}\xi\big)\hat{\Phi} v(\xi)\big|^2\,d\xi \\
&\leqslant \int_{\mathbb{R}} \frac{(1+|\xi|^2)^q}{(1+|\xi|^2)^s}\big|\hat{\rho}\big(\varepsilon^{\frac{1}{4}}\xi\big)\big|^2\big(1+|\xi|^2\big)^s\big|\hat{\Phi} v(\xi)\big|^2\,d\xi \\
&\leqslant \|\Phi v\|^2_{H^s}\sup_{\xi\in\mathbb{R}}\bigg[\frac{(1+|\xi|^2)^q}{(1+|\xi|^2)^s}\big|\hat{\rho}\big(\varepsilon^{\frac{1}{4}}\xi\big)\big|^2\bigg].
\end{aligned}$$

When $q \leqslant s$, we get

$$\sup_{\xi\in\mathbb{R}}\bigg[\frac{(1+|\xi|^2)^q}{(1+|\xi|^2)^s}\big|\hat{\rho}\big(\varepsilon^{\frac{1}{4}}\xi\big)\big|^2\bigg] \leqslant \sup_{\xi\in\mathbb{R}}\big|\hat{\rho}\big(\varepsilon^{\frac{1}{4}}\xi\big)\big|^2 \leqslant c,$$

which proves (4.21). If $q > s$, it has

$$\begin{aligned}
\sup_{\xi\in\mathbb{R}}\bigg[\frac{(1+|\xi|^2)^q}{(1+|\xi|^2)^s}\big|\hat{\rho}\big(\varepsilon^{\frac{1}{4}}\xi\big)\big|^2\bigg] &\leqslant \sup_{\varepsilon^{\frac{1}{4}}\xi=K\in\mathbb{R}}\bigg[\frac{(1+|\varepsilon^{-\frac{1}{4}}K|^2)^q}{(1+|\varepsilon^{-\frac{1}{4}}K|^2)^s}\big|\hat{\rho}(K)\big|^2\bigg] \\
&\leqslant \varepsilon^{\frac{s-q}{2}}\sup_{K\in\mathbb{R}}\Big[\big(\varepsilon^{\frac{1}{2}}+|K|^2\big)^{q-s}\big|\hat{\rho}(K)\big|^2\Big] \leqslant c\varepsilon^{\frac{s-q}{2}},
\end{aligned}$$

which proves (4.22). For $q \leqslant s$, we have

$$
\begin{aligned}
\left\|(\Phi_\varepsilon - \Phi)v\right\|_{H^q}^2 &= \int_{\mathbb{R}} \left(1+|\xi|^2\right)^q \left|\hat{\rho}\left(\varepsilon^{\frac{1}{4}}\xi\right)\hat{\Phi v}(\xi) - \hat{\Phi v}(\xi)\right|^2 d\xi \\
&\leqslant \int_{\mathbb{R}} \frac{(1+|\xi|^2)^q}{(1+|\xi|^2)^s}\left(1+|\xi|^2\right)^s \left|\hat{\Phi v}(\xi)\right|^2 \left|\hat{\rho}\left(\varepsilon^{\frac{1}{4}}\xi\right)-1\right|^2 d\xi \\
&\leqslant \|\Phi v\|_{H^s}^2 \sup_{\xi\in\mathbb{R}}\left[\frac{(1+|\xi|^2)^q}{(1+|\xi|^2)^s}\left|\hat{\rho}\left(\varepsilon^{\frac{1}{4}}\xi\right)-1\right|^2\right] \\
&\leqslant \|\Phi v\|_{H^s}^2 \varepsilon^{\frac{s-q}{2}} \sup_{\varepsilon^{\frac{1}{4}}\xi = K\in\mathbb{R}} \left[\left(\varepsilon^{\frac{1}{2}}+|K|^2\right)^{q-s}\left|\hat{\rho}(K)-1\right|^2\right] \\
&\leqslant c\varepsilon^{\frac{s-q}{2}},
\end{aligned}
$$

which results (4.23). The expression (4.24) is a common result since Φ_ε uniformly converges to Φ in space $L_2^{0,s}$ with $s > 0$. $\square$

Lemma 4.7. *Let $u(t,x) = u_\varepsilon(t,x)$ be a solution of the problem* (4.15)–(4.16)*, then there exist positive constants c_1, c_2, and M, which are independent of ε, such that the inequalities*

$$
\mathbb{E}\sup_{t\in[0,T]}\|u\|_{H^s}^2 \leqslant \frac{c_1 + T\|\Phi_\varepsilon\|_{L_2^{0,s-1}}^2}{(1-MT)^{c_2}}, \tag{4.25}
$$

$$
\mathbb{E}\sup_{t\in[0,T]}\|u\|_{H^{s+k}}^2 \leqslant \frac{c_1\varepsilon^{-\frac{k}{2}} + T\|\Phi_\varepsilon\|_{L_2^{0,s+k-1}}^2}{(1-MT)^{c_2}} \tag{4.26}
$$

hold for any sufficiently small ε and $T < \frac{1}{M}$ and $s > 3/2$.

Proof. Let r be a real number with $3/2 < r < s$. Letting $q = r-1$ in inequality (4.4) and using $\|u_x\|_{L^\infty} \lesssim \|u\|_{H^r}$, we have

$$
\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}\|u\|_{H^r}^2 &\lesssim \mathbb{E}\|u_0\|_{H^r}^2 + T|\Phi_\varepsilon|_{L_2^{0,r-1}}^2 + \mathbb{E}\int_0^T \|u\|_{H^r}^3\,ds \\
&:= A + \mathbb{E}\int_0^T \|u\|_{H^r}^3\,ds := Y(T).
\end{aligned} \tag{4.27}
$$

From (4.27), we obtain

$$
\frac{dY(T)}{dT} \leqslant \mathbb{E}\sup_{t\in[0,T]}\|u\|_{H^r}^3 \leqslant CY(T)^{3/2},
$$

which derives $Y \leqslant \frac{A}{(1-C\sqrt{A}T)^2}$ for $T < \frac{1}{C\sqrt{A}}$, i.e.

$$
\mathbb{E}\sup_{t\in[0,T]}\|u\|_{H^r}^2 \leqslant \frac{A}{(1-C\sqrt{A}T)^2} \leqslant \frac{M}{(1-MT)^2}, \tag{4.28}
$$

for $T < \frac{1}{M}$, where

$$A = \mathbb{E}\|u_{0\varepsilon}\|_{H^r}^2 + T\|\Phi_\varepsilon\|_{L_2^{0,r-1}}^2 \quad \text{and} \quad M = \max\{A, C\sqrt{A}\}.$$

Using (4.28), $\|u_x\|_{L^\infty} \leqslant C\|u\|_{H^r}$, $r > 3/2$ and (4.4) with $q + 1 = s$, one can obtain

$$\mathbb{E}\sup_{t\in[0,T]}\|u\|_{H^s}^2 \leqslant c + T\|\Phi_\varepsilon\|_{L_2^{0,s-1}}^2 + C\mathbb{E}\int_0^T \frac{\sqrt{M}}{1-Ms}\|u\|_{H^s}^2\,ds, \tag{4.29}$$

for $T \in [0, \frac{1}{M})$. From Gronwall's inequality, one has

$$\mathbb{E}\sup_{t\in[0,T]}\|u\|_{H^s}^2 \leqslant \frac{c_1 + T\|\Phi_\varepsilon\|_{L_2^{0,s-1}}^2}{(1-MT)^{c_2}}, \tag{4.30}$$

with $T < \frac{1}{M}$.

In a similar manner, for $q + 1 = s + k$ in (4.4), one can obtain the estimate

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\|u\|_{H^{s+k}}^2 \leqslant{}& \mathbb{E}\|u_0\|_{H^{s+k-1}}^2 + \mathbb{E}\|u_0\|_{H^{s+k}}^2\\ &+ T|\Phi_\varepsilon|_{L_2^{0,s+k-1}}^2 + C\mathbb{E}\int_0^T \frac{\sqrt{M}}{1-Ms}\|u\|_{H^{s+k}}^2\,ds,\end{aligned} \tag{4.31}$$

from Gronwall's inequality, (4.17) and (4.18), we have

$$\mathbb{E}\sup_{t\in[0,T]}\|u\|_{H^{s+k}}^2 \leqslant \frac{c_1\varepsilon^{-\frac{k}{2}} + T\|\Phi_\varepsilon\|_{L_2^{0,s+k-1}}^2}{(1-Mt)^{c_2}}, \tag{4.32}$$

with $T < \frac{1}{M}$.

Now, we prove that u_ε is a Cauchy sequence. Let u_ε and u_δ be solutions of problem (4.15)–(4.16), corresponding to the parameters ε and δ, with $0 < \varepsilon < \delta < 1/4$, and let $w = u_\varepsilon - u_\delta$. Then w satisfies the problem

$$\begin{aligned}&w_t - \varepsilon\partial_x^3 u_\varepsilon + \delta\partial_x^3 u_\delta + \frac{1}{2}\partial_x\big[w(u_\varepsilon + u_\delta)\big] + \big(1-\partial_x^2\big)^{-1}\partial_x\left[w(u_\varepsilon + u_\delta) + \frac{1}{2}w_x(u_\varepsilon + u_\delta)_x\right]\\ &\quad = \big(1-\partial_x^2\big)^{-1}(\Phi_\varepsilon - \Phi_\delta)\,dW,\end{aligned} \tag{4.33}$$

$$w(x,0) = w_{0\varepsilon,\delta}(x) = u_{0\varepsilon}(x) - u_{0\delta}(x), \quad x \in \mathbb{R},\ t > 0. \qquad \square \tag{4.34}$$

Proposition 4.1. *For a.e. $\omega \in \Omega$, there exists $T = T_\omega > 0$ such that u_ε, the solution of (4.15)–(4.16), is a Cauchy sequence in the space $L^2(\Omega; C([0,T]; H^s(\mathbb{R})))$ for $s > 3/2$.*

Proof. For $1/2 < q < \min(1, s-1)$, applying the Itô formula on $\|w\|_{H^q}^2$, we get

$$\begin{aligned}
\|w\|_{H^q}^2 = {} & \|w_{0\varepsilon,\delta}\|_{H^q}^2 + 2\varepsilon \int_0^t \int_{\mathbb{R}} \Lambda^q w \Lambda^q \partial_x^3 u_\varepsilon \,dx\,ds - 2\delta \int_0^t \int_{\mathbb{R}} \Lambda^q w \Lambda^q \partial_x^3 u_\delta \,dx\,ds \\
& - \int_0^t \int_{\mathbb{R}} \Lambda^q w \Lambda^q \big(w(u_\varepsilon + u_\delta)\big)_x \,dx\,ds - 2\int_0^t \int_{\mathbb{R}} \Lambda^q w \Lambda^{q-2} \big(w(u_\varepsilon + u_\delta)\big)_x \,dx\,ds \\
& - \int_0^t \int_{\mathbb{R}} \Lambda^q w \Lambda^{q-2} \big(w_x(u_{\varepsilon x} + u_{\delta x})\big)_x \,dx\,ds + \int_0^t \|\Phi_\varepsilon - \Phi_\delta\|_{L_2^{0,q-1}}^2 \,ds \\
& + 2\int_0^t \int_{\mathbb{R}} \Lambda^q w \Lambda^{q-2} (\Phi_\varepsilon - \Phi_\delta)\,dW\,dx.
\end{aligned} \tag{4.35}$$

Using Hölder's inequality, Young's inequality and (4.25), we have

$$\begin{aligned}
2\varepsilon \mathbb{E} \sup_{t\in[0,T]} \int_0^t \int_{\mathbb{R}} \Lambda^q w \Lambda^q \partial_x^3 u_\varepsilon \,dx\,ds & \leqslant 2\varepsilon \mathbb{E} \sup_{t\in[0,T]} \int_0^t \|w\|_{H^q} \|u_\varepsilon\|_{H^{q+3}} \,ds \\
& \leqslant \mathbb{E} \sup_{t\in[0,T]} \int_0^t \|w\|_{H^q}^2 \,ds + \varepsilon^2 \mathbb{E} \sup_{t\in[0,T]} \int_0^t \|u_\varepsilon\|_{H^{q+3}}^2 \,ds \\
& \leqslant C\varepsilon + \mathbb{E} \int_0^T \|w\|_{H^q}^2 \,ds,
\end{aligned} \tag{4.36}$$

similar to (4.36), we have

$$2\delta \mathbb{E} \sup_{t\in[0,T]} \int_0^t \int_{\mathbb{R}} \Lambda^q w \Lambda^q \partial_x^3 u_\delta \,dx\,ds \leqslant C\delta + \mathbb{E} \int_0^T \|w\|_{H^q}^2 \,ds. \tag{4.37}$$

From Lemma 4.3 and (4.25), we obtain

$$\begin{aligned}
\mathbb{E} \sup_{t\in[0,T]} \int_0^t \int_{\mathbb{R}} \Lambda^q w \Lambda^q \big(w(u_\varepsilon + u_\delta)\big)_x \,dx\,ds & \lesssim \mathbb{E} \sup_{t\in[0,T]} \int_0^t \|w\|_{H^q}^2 \big\|(u_\varepsilon + u_\delta)\big\|_{H^{q+1}} \,ds \\
& \lesssim \mathbb{E} \sup_{t\in[0,T]} \int_0^t \|w\|_{H^q}^2 \big(\|u_\varepsilon\|_{H^{q+1}} + \|u_\delta\|_{H^{q+1}}\big) \,ds \\
& \lesssim \mathbb{E} \int_0^T \|w\|_{H^q}^2 \,ds.
\end{aligned} \tag{4.38}$$

Moreover, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \Lambda^q w \Lambda^{q-2}\big(w(u_\varepsilon + u_\delta)\big)_x\,dx \\
&\quad \lesssim \|w\|_{H^q}\big\|\Lambda^{q-2}\big(w(u_\varepsilon+u_\delta)\big)_x\big\|_{L^2} \\
&\quad \lesssim \|w\|_{H^q}\bigg(\int_{\mathbb{R}}\bigg(\int_{\mathbb{R}}\big(1+\xi^2\big)^{\frac{q-1}{2}}\hat{w}(\xi-\eta)(\hat{u}_\varepsilon+\hat{u}_\delta)(\eta)\,d\eta\bigg)^2 d\xi\bigg)^{1/2} \\
&\quad \lesssim \|w\|_{H^q}\bigg(\int_{\mathbb{R}}\frac{(\int_{\mathbb{R}}[(1+(\xi-\eta)^2)^{\frac{q}{2}}+(1+\eta^2)^{\frac{q}{2}}]|\hat{w}(\xi-\eta)(\hat{u}_\varepsilon+\hat{u}_\delta)(\eta)|\,d\eta)^2}{1+\xi^2}\,d\xi\bigg)^{1/2} \\
&\quad \lesssim \|w\|_{H^q}\big(\|w\|_{H^q}\|u_\varepsilon+u_\delta\|_{L^2}+\|w\|_{L^2}\|u_\varepsilon+u_\delta\|_{H^q}\big),
\end{aligned} \tag{4.39}$$

so, using Hölder's inequality, (4.25) and the imbedding theorem, we have

$$\mathbb{E}\sup_{t\in[0,T]}\int_0^t\int_{\mathbb{R}}\Lambda^q w\Lambda^{q-2}\big(w(u_\varepsilon+u_\delta)\big)_x\,dx\,ds \lesssim \mathbb{E}\int_0^T\|w\|_{H^q}^2\,ds. \tag{4.40}$$

Using the following inequality from Lemma 3.1.1 in [3] ($B>0$ is a constant),

$$\int_{\mathbb{R}}\frac{d\xi}{(1+(\xi-\eta)^2)^{s-1}(1+\xi^2)^{1-q}} \leqslant \frac{B}{(1+\eta^2)^{1-q}},$$

one may obtain the estimate

$$\begin{aligned}
&\int_{\mathbb{R}}\Lambda^q w\Lambda^{q-2}\big(w_{1x}(u_{\varepsilon x}+u_{\delta x})\big)_x\,dx\,ds \\
&\quad = \int_{\mathbb{R}}\big(1+\xi^2\big)^{q-1}\xi\hat{w}(\xi)\,d\xi\int_{\mathbb{R}}(\xi-\eta)\hat{w}(\xi-\eta)\eta(\hat{u}_\varepsilon+\hat{u}_\delta)(\eta)\,d\eta \\
&\quad \lesssim \int_{\mathbb{R}}\big(1+\xi^2\big)^{q-1}\big|\xi\hat{w}(\xi)\big|\,d\xi \\
&\qquad \times\bigg(\int_{\mathbb{R}}\frac{|\xi-\eta|^2|\hat{w}(\xi-\eta)|^2}{(1+\eta^2)^{s-1}}\,d\eta\int_{\mathbb{R}}|\eta|^2\big(1+\eta^2\big)^{s-1}\big|(\hat{u}_\varepsilon+\hat{u}_\delta)(\eta)\big|^2\,d\eta\bigg)^{1/2} \\
&\quad \lesssim \|u_\varepsilon+u_\delta\|_{H^s}\bigg(\int_{\mathbb{R}}\big(1+\xi^2\big)^q\big|\hat{w}(\xi)\big|^2\,d\xi\bigg)^{1/2}\bigg(\int_{\mathbb{R}}\frac{|\xi|^2}{(1+\xi^2)^{2-q}}\,d\xi\int_{\mathbb{R}}\frac{|\eta|^2|\hat{w}(\xi)|^2}{(1+(\xi-\eta)^2)^{s-1}}\,d\eta\bigg)^{1/2} \\
&\quad \lesssim \|u_\varepsilon+u_\delta\|_{H^s}\|w\|_{H^q}\bigg(\int_{\mathbb{R}}|\eta|^2\big|\hat{w}(\eta)\big|^2\,d\eta\int_{\mathbb{R}}\frac{d\xi}{(1+(\xi-\eta)^2)^{s-1}(1+\xi^2)^{1-q}}\bigg)^{1/2} \\
&\quad \lesssim \|u_\varepsilon+u_\delta\|_{H^s}\|w\|_{H^q}^2,
\end{aligned}$$

so, using (4.25), we have

$$\mathbb{E}\sup_{t\in[0,T]}\int_0^t\int_{\mathbb{R}}\Lambda^q w\Lambda^{q-2}\big(w_{1x}(u_{\varepsilon x}+u_{\delta x})\big)_x\,dx\,ds\lesssim\mathbb{E}\int_0^T\|w\|_{H^q}^2\,ds. \tag{4.41}$$

By the B–D–G inequality and (4.23), we obtain

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}\int_0^t\int_{\mathbb{R}}\Lambda^q w\Lambda^{q-2}(\Phi_\varepsilon-\Phi_\delta)\,dW\,dx&\leqslant 3\mathbb{E}\left\{\int_0^T\int_{\mathbb{R}}\left|\Lambda^{q-1}w\Lambda^{q-1}(\Phi_\varepsilon-\Phi_\delta)\right|^2dx\,dt\right\}^{1/2}\\&\leqslant 3\mathbb{E}\left\{\sup_{t\in[0,T]}\|w\|_{H^{q-1}}\int_0^T\|\Phi_\varepsilon-\Phi_\delta\|_{L_2^{0,q-1}}^2\,dt\right\}^{1/2}\\&\leqslant\frac{1}{4}\mathbb{E}\sup_{t\in[0,T]}\|w\|_{H^{q-1}}^2+9T\|\Phi_\varepsilon-\Phi_\delta\|_{L_2^{0,q-1}}^2\\&\leqslant\frac{1}{4}\mathbb{E}\sup_{t\in[0,T]}\|w\|_{H^{q-1}}^2+C(\varepsilon+\delta).\end{aligned} \tag{4.42}$$

From (4.35)–(4.42), we have

$$\mathbb{E}\sup_{t\in[0,T]}\|w\|_{H^q}^2\lesssim\|w_{0\varepsilon,\delta}\|_{H^q}^2+(\varepsilon+\delta)+\mathbb{E}\int_0^T\|w\|_{H^q}^2\,ds. \tag{4.43}$$

Applying Gronwall's inequality to (4.43) together with (4.19), it yields

$$\mathbb{E}\sup_{t\in[0,T]}\|w\|_{H^q}^2\lesssim(\varepsilon+\delta)e^{Ct}. \tag{4.44}$$

Using the Itô formula on $\|w\|_{H^s}^2$, we have the equation

$$\begin{aligned}\|w\|_{H^s}^2=\|w_{0\varepsilon,\delta}\|_{H^s}^2&+2\varepsilon\int_0^t\int_{\mathbb{R}}\Lambda^s w\Lambda^s\partial_x^3u_\varepsilon\,dx\,ds-2\delta\int_0^t\int_{\mathbb{R}}\Lambda^s w\Lambda^q\partial_x^3u_\delta\,dx\,ds\\&-\int_0^t\int_{\mathbb{R}}\Lambda^s w\Lambda^s\big(w(u_\varepsilon+u_\delta)\big)_x\,dx\,ds-2\int_0^t\int_{\mathbb{R}}\Lambda^s w\Lambda^{s-2}\big(w(u_\varepsilon+u_\delta)\big)_x\,dx\,ds\\&-\int_0^t\int_{\mathbb{R}}\Lambda^s w\Lambda^{s-2}\big(w_x(u_{\varepsilon x}+u_{\delta x})\big)_x\,dx\,ds+\int_0^t\|\Phi_\varepsilon-\Phi_\delta\|_{L_2^{0,s-1}}^2\,ds\\&+2\int_0^t\int_{\mathbb{R}}\Lambda^s w\Lambda^{s-2}(\Phi_\varepsilon-\Phi_\delta)\,dW\,dx.\end{aligned} \tag{4.45}$$

Using Hölder's inequality, Young's inequality and (4.26), we have

$$
\begin{aligned}
2\varepsilon\mathbb{E}\sup_{t\in[0,T]}\int_0^t\int_{\mathbb{R}}\Lambda^s w\Lambda^s\partial_x^3 u_\varepsilon\,dx\,ds &\leqslant 2\varepsilon\mathbb{E}\sup_{t\in[0,T]}\int_0^t\|w\|_{H^s}\|u_\varepsilon\|_{H^{s+3}}\,ds\\
&\leqslant \mathbb{E}\sup_{t\in[0,T]}\int_0^t\|w\|_{H^s}^2\,ds+\varepsilon^2\mathbb{E}\sup_{t\in[0,T]}\int_0^t\|u_\varepsilon\|_{H^{s+3}}^2\,ds\\
&\leqslant C\varepsilon+\mathbb{E}\int_0^T\|w\|_{H^s}^2\,ds,
\end{aligned}
\tag{4.46}
$$

similar to (4.46), we have

$$
2\delta\mathbb{E}\sup_{t\in[0,T]}\int_0^t\int_{\mathbb{R}}\Lambda^s w\Lambda^s\partial_x^3 u_\delta\,dx\,ds\leqslant C\delta+\frac{1}{2}\mathbb{E}\int_0^T\|w\|_{H^s}^2\,ds. \tag{4.47}
$$

Using Lemma 4.3, (4.25)–(4.26), (4.44), $\|u\|_{L^\infty}\lesssim\|u\|_{H^q}$, $\|u_x\|_{L^\infty}\lesssim\|u\|_{H^s}$ and the Cauchy inequality, we have

$$
\begin{aligned}
&\mathbb{E}\sup_{t\in[0,T]}\int_0^t\int_{\mathbb{R}}\Lambda^s w\Lambda^s\big(w(u_\varepsilon+u_\delta)_x\big)\,dx\,ds\\
&=\mathbb{E}\sup_{t\in[0,T]}\int_0^t\int_{\mathbb{R}}\Lambda^s w\Lambda^s\big(w^2+2wu_\delta\big)_x\,dx\,ds\\
&\lesssim\mathbb{E}\sup_{t\in[0,T]}\int_0^t\big(\|w_x\|_{L^\infty}+\|u_\delta\|_{H^s}\big)\|w\|_{H^s}^2+\|u_\delta\|_{H^{s+1}}\|w\|_{H^q}\|w\|_{H^s}\,ds\\
&\lesssim\mathbb{E}\int_0^T\|w\|_{H^s}^2+\|u_\delta\|_{H^{s+1}}^2\|w\|_{H^q}^2\,ds\\
&\lesssim\mathbb{E}\int_0^T\|w\|_{H^s}^2\,ds+\big(\varepsilon^{\frac{1}{2}}+\delta^{\frac{1}{2}}\big).
\end{aligned}
\tag{4.48}
$$

Moreover, we have

$$
\begin{aligned}
&\int_0^t\int_{\mathbb{R}}\Lambda^s w\Lambda^{s-2}\big(w(u_\varepsilon+u_\delta)\big)_x\,dx\\
&\quad\lesssim\|w\|_{H^s}\big\|\Lambda^{s-2}\big(w(u_\varepsilon+u_\delta)\big)_x\big\|_{L^2}
\end{aligned}
$$

$$\lesssim \|w\|_{H^s}\bigg(\int_{\mathbb{R}}\bigg(\int_{\mathbb{R}}\big(1+\xi^2\big)^{\frac{s-1}{2}}\hat{w}(\xi-\eta)(\hat{u}_\varepsilon+\hat{u}_\delta)(\eta)\,d\eta\bigg)^2 d\xi\bigg)^{1/2}$$

$$\lesssim \|w\|_{H^s}\bigg(\int_{\mathbb{R}}\frac{(\int_{\mathbb{R}}[(1+(\xi-\eta)^2)^{\frac{s}{2}}+(1+\eta^2)^{\frac{s}{2}}]|\hat{w}(\xi-\eta)(\hat{u}_\varepsilon+\hat{u}_\delta)(\eta)|\,d\eta)^2}{1+\xi^2}\,d\xi\bigg)^{1/2}$$

$$\lesssim \|w\|_{H^s}\big(\|w\|_{H^s}\|u_\varepsilon+u_\delta\|_{L^2}+\|w\|_{L^2}\|u_\varepsilon+u_\delta\|_{H^s}\big),$$

so, using (4.25) and the imbedding theorem,

$$\mathbb{E}\sup_{t\in[0,T]}\int_0^t\int_{\mathbb{R}}\Lambda^s w\,\Lambda^{s-2}\big(w(u_\varepsilon+u_\delta)\big)_x\,dx\,ds\lesssim \mathbb{E}\int_0^T\|w\|_{H^s}^2\,ds. \tag{4.49}$$

In addition, we have

$$\begin{aligned}
&\int_{\mathbb{R}}\Lambda^s w\,\Lambda^{s-2}\big(w_x(u_{\varepsilon x}+u_{\delta x})\big)_x\,dx\,ds\\
&\quad=\int_{\mathbb{R}}\big(1+\xi^2\big)^{s-1}\xi\,\hat{w}(\xi)\,d\xi\int_{\mathbb{R}}(\xi-\eta)\hat{w}(\xi-\eta)\eta(\hat{u_\varepsilon}+\hat{u_\delta})(\eta)\,d\eta\\
&\quad\lesssim\int_{\mathbb{R}}\big(1+\xi^2\big)^{s/2}\big|\hat{w}(\xi)\big|\,d\xi\int_{\mathbb{R}}\Big[\big(1+(\xi-\eta)^2\big)^{\frac{s-1}{2}}+\big(1+\eta^2\big)^{\frac{s-1}{2}}\Big]\\
&\qquad\cdot\big|(\xi-\eta)\hat{w}(\xi-\eta)\eta(\hat{u_\varepsilon}+\hat{u_\delta})(\eta)\big|\,d\eta\\
&\quad\lesssim\|w\|_{H^s}\big(\|\hat{w}_{1x}\|_{L^1}\|u_\varepsilon+u_\delta\|_{H^s}+\|w\|_{H^s}\big\|(\hat{u_\varepsilon}+\hat{u_\delta})_x\big\|_{L^1}\big),
\end{aligned}$$

so, using (4.25) and the imbedding theorem,

$$\mathbb{E}\sup_{t\in[0,T]}\int_0^t\int_{\mathbb{R}}\Lambda^s w\,\Lambda^{s-2}\big(w_x(u_{\varepsilon x}+u_{\delta x})\big)_x\,dx\,ds\lesssim \mathbb{E}\int_0^T\|w\|_{H^s}^2\,ds. \tag{4.50}$$

By the B–D–G inequality, we obtain

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]}\int_0^t\int_{\mathbb{R}}\Lambda^s w\,\Lambda^{s-2}(\Phi_\varepsilon-\Phi_\delta)\,dW\,dx&\leqslant 3\mathbb{E}\bigg\{\int_0^T\int_{\mathbb{R}}\big|\Lambda^{s-1}w\,\Lambda^{s-1}(\Phi_\varepsilon-\Phi_\delta)\big|^2\,dx\,dt\bigg\}^{1/2}\\
&\leqslant 3\mathbb{E}\bigg\{\sup_{t\in[0,T]}\|w\|_{H^{s-1}}\int_0^T\big|\Lambda^{s-1}(\Phi_\varepsilon-\Phi_\delta)\big|^2\,dt\bigg\}^{1/2}\\
&\leqslant\frac{1}{4}\mathbb{E}\sup_{t\in[0,T]}\|w\|_{H^{s-1}}^2+9T\|\Phi_\varepsilon-\Phi_\delta\|_{L_2^{0,s-1}}.
\end{aligned}\tag{4.51}$$

It follows from (4.45)–(4.51), that we have

$$\mathbb{E}\sup_{t\in[0,T]}\|w\|_{H^s}^2 \lesssim \|w_{0\varepsilon,\delta}\|_{H^s}^2 + \left(\varepsilon^{\frac{1}{2}}+\delta^{\frac{1}{2}}\right) + T\|\Phi_\varepsilon-\Phi_\delta\|_{L_2^{0,s-1}} + \mathbb{E}\int_0^T \|w\|_{H^s}^2\,dt. \tag{4.52}$$

It follows from Gronwall's inequality that

$$\mathbb{E}\sup_{t\in[0,T]}\|w\|_{H^s}^2 \lesssim \left[\|w_{0\varepsilon,\delta}\|_{H^s}^2 + \left(\varepsilon^{\frac{1}{2}}+\delta^{\frac{1}{2}}\right) + T\|\Phi_\varepsilon-\Phi_\delta\|_{L_2^{0,s-1}}\right]e^{Ct}. \tag{4.53}$$

Then from (4.20), (4.24), we have

$$\mathbb{E}\sup_{t\in[0,T]}\|w\|_{H^s}^2 \to 0 \quad \text{as } \varepsilon\to 0,\ \delta\to 0, \tag{4.54}$$

which completes the proof of Proposition 4.1. □

From Proposition 4.1, we can easily get the local well-posedness of (1.1)–(1.2).

Theorem 4.1. *Assume that for some s with $s>3/2$, $\Phi\in L_2^{0,s-1}$. Let $u_0(\omega,x)\in H^s(\mathbb{R})$ for a.e. $\omega\in\Omega$ and u_0 being $\mathcal{F}_0$-measurable. Then for a.e. $\omega\in\Omega$, there exists $T=T_\omega>0$ such that the Cauchy problem* (1.1)–(1.2) *has a unique solution $u\in L^2(\Omega;C([0,T];H^s(\mathbb{R})))$.*

Proof. Since u_ε is a Cauchy sequence in the space $L^2(\Omega;C([0,T];H^s(\mathbb{R})))$, there exists a limit $u(x,t)$ of the sequence as $\varepsilon\to 0$. Taking the limit on both sides of (4.15)–(4.16) as $\varepsilon\to 0$, we have that $u\in L^2(\Omega;C([0,T];H^s(\mathbb{R})))$ is the solution of the problem

$$\partial_t u + \partial_x^2\partial_t u + 3u\partial_x u - 2\partial_x u\partial_x^2 u - u\partial_x^3 u = \Phi\frac{\partial^2 B}{\partial t\partial x},$$
$$u(x,0)=u_0(x),\quad x\in\mathbb{R},\ t>0,$$

which completes the proof of existence.

For the uniqueness, let u, v be two solutions of (1.1)–(1.2) in the space $L^2(\Omega;C([0,T];H^s(\mathbb{R})))$. Then $w=u-v$ satisfies the Cauchy problem

$$\partial_t w + \frac{1}{2}\partial_x\left(u^2-v^2\right) + \left(1-\partial_x^2\right)^{-1}\partial_x\left[\left(u^2-v^2\right)+\frac{1}{2}\left((\partial_x u)^2-(\partial_x v)^2\right)\right]=0,$$
$$w(x,0)=0,\quad x\in\mathbb{R},\ t>0,$$

which, as in the deterministic case, applying the operator $\Lambda^q w\Lambda^q$ with $1/2<q<\min\{1,s-1\}$ to both sides of the above equation and then integrating with respect to x, obtains the equality

$$\frac{1}{2}\frac{d}{dt}\|w\|_{H^q}^2 = -\frac{1}{2}\int_{\mathbb{R}}\Lambda^q w\Lambda^q\left(w(u+v)\right)_x dx - \int_{\mathbb{R}}\Lambda^q w\Lambda^{q-2}\partial_x\left[w(u+v)+\frac{1}{2}w_x(u_x+v_x)\right]dx.$$

It follows from Lemma 4.3, (4.38)–(4.41) that there is a constant C such that

$$\mathbb{E}\|w\|_{H^q}^2 \leqslant C\mathbb{E}\int_0^t \|w\|_{H^q}^2\,dt.$$

Then Gronwall's inequality leads to the conclusion that

$$\mathbb{E}\sup_{t\in[0,T]}\|w\|_{H^q}^2=0,$$

which derives $u=v$ a.s. This completes the proof. $\square$

5. Blow-up solution

In this section, we shall verify that there are solutions of the stochastic Camassa–Holm equation, whose H^q-norms blow up in finite time for any $q>3/2$.

Theorem 5.1. *Assume $\Phi\in L_2^{0,s-1}$, $u_0\in H^s(\mathbb{R})$ for a.e. $\omega\in\Omega$ is $\mathcal{F}_0$-measurable for $s>3/2$ and satisfies the conditions*

$$\mathbb{E}\int_{\mathbb{R}}u_{0x}^3\,dx<0,\qquad \frac{8}{3}b\big(\mathbb{E}\|u_0\|_{H^1}^2+T\|\Phi\|_{L_2^{0,0}}\big)\leqslant-\mathbb{E}\int_{\mathbb{R}}u_{0x}^3\,dx,$$

where $b=C(\mathbb{E}\|u_0\|_{H^1},T,\|\Phi\|_{L_2^{0,0}})$ is a constant to be specified in the proof, then there is a $0<T^*\leqslant -\frac{4}{3\mathbb{E}\int_{\mathbb{R}}u_{0x}^3\,dx}(\mathbb{E}\|u_0\|_{H^1}^2+T\|\Phi\|_{L_2^{0,0}})$ such that the corresponding solution u blows up in $H^s(\mathbb{R})$ at the time T^* in the sense that

$$\lim_{t\to T^*}\sup\mathbb{E}\|u_x\|_{L^\infty}=\infty,\qquad \lim_{t\to T^*}\mathbb{E}\|u_x\|_{H^q}=\infty,$$

for any $q\in(3/2,s]$.

Proof. It follows from Theorem 4.1 that there exists a $T>0$ such that the Cauchy problem (1.1)–(1.2) has a unique solution $u\in L^2(\Omega;C([0,T];H^s(\mathbb{R})))$.

Using the Itô formula on $\int_{\mathbb{R}}u_x^3\,dx$ after a regularization argument as in (4.15)–(4.16), we can obtain

$$\begin{aligned}
\int_{\mathbb{R}}u_x^3\,dx&=\int_{\mathbb{R}}u_{0x}^3\,dx-\frac{3}{2}\int_0^t\int_{\mathbb{R}}\partial_x^2u^2u_x^2\,dx\,ds+3\int_0^t\int_{\mathbb{R}}\left(u^2+\frac{1}{2}(\partial_xu)^2\right)u_x^2\,dx\,ds\\
&\quad-3\int_0^t\int_{\mathbb{R}}\left(1-\partial_x^2\right)^{-1}\left[u^2+\frac{1}{2}(\partial_xu)^2\right]u_x^2\,dx\,ds\\
&\quad+3\int_0^t\int_{\mathbb{R}}\left(1-\partial_x^2\right)^{-1}\partial_x\Phi u_x^2\,dW\,dx+3\int_0^t\int_{\mathbb{R}}u_x\|\Phi\|_{L_2^{0,-1}}\,dx\,ds\\
&=\|u_{0x}\|_{L^3}^3-\frac{3}{2}\int_0^t\int_{\mathbb{R}}u_x^4\,dx\,ds+6\int_0^t\int_{\mathbb{R}}u_x^2\,dx\,ds\\
&\quad-3\int_0^t\int_{\mathbb{R}}\left(1-\partial_x^2\right)^{-1}\left[u^2+\frac{1}{2}(\partial_xu)^2\right]u_x^2\,dx\,ds
\end{aligned}$$

$$+3\int_0^t\int_{\mathbb{R}}\left(1-\partial_x^2\right)^{-1}\partial_x\Phi u_x^2\,dW\,dx+3\int_0^t\int_{\mathbb{R}}u_x\|\Phi\|_{L_2^{0,-1}}\,dx\,ds,$$

which implies the following equality

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}}u_x^3\,dx+\frac{3}{2}\mathbb{E}\int_0^t\int_{\mathbb{R}}u_x^4\,dx\,ds\\
&\quad=\mathbb{E}\int_{\mathbb{R}}u_{0x}^3\,dx+6\mathbb{E}\int_0^t\int_{\mathbb{R}}u_x^2\,dx\,ds-3\mathbb{E}\int_0^t\int_{\mathbb{R}}\left(1-\partial_x^2\right)^{-1}\left[u^2+\frac{1}{2}(\partial_x u)^2\right]u_x^2\,dx\,ds\\
&\qquad+3\mathbb{E}\int_0^t\|\Phi\|_{L_2^{0,-1}}\int_{\mathbb{R}}u_x\,dx\,ds.
\end{aligned}\tag{5.1}$$

Since $\mathbb{E}|\int_{\mathbb{R}}u_x^3\,dx|\leqslant\mathbb{E}(\int_{\mathbb{R}}|u_x|^4\,dx)^{1/2}(\int_{\mathbb{R}}|u_x|^2\,dx)^{1/2}$, it follows that

$$\mathbb{E}\int_{\mathbb{R}}|u_x|^4\,dx\geqslant\frac{1}{\mathbb{E}\|u\|_{H^1}^2}\left(\mathbb{E}\int_{\mathbb{R}}u_x^3\,dx\right)^2.\tag{5.2}$$

Since for $f\in H^1$, $|(1-\partial_x^2)^{-1}f(x)|=|\frac{1}{2}\int_{\mathbb{R}}e^{-|x-y|}f(y)\,dy|\leqslant\frac{1}{2}\|f\|_{H^1}$ or $\leqslant\frac{1}{2}\int_{\mathbb{R}}|f(y)|\,dy$, it follows

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}}\left(1-\partial_x^2\right)^{-1}\left[u^2+\frac{1}{2}(\partial_x u)^2\right]u_x^2\,dx\\
&\quad\leqslant\mathbb{E}\|u\|_{H^1}^2\left|\left(1-\partial_x^2\right)^{-1}\left[u^2+\frac{1}{2}(\partial_x u)^2\right]\right|_{L^\infty}\leqslant\frac{3}{4}\mathbb{E}\|u\|_{H^1}^4.
\end{aligned}\tag{5.3}$$

Applying (5.2), (5.3) and the inequality (4.3) to (5.1), it leads to the estimate

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}}u_x^3\,dx+\frac{3}{2}\int_0^t\frac{1}{\mathbb{E}\|u\|_{H^1}^2}\left(\mathbb{E}\int_{\mathbb{R}}u_x^3\,dx\right)^2ds\\
&\quad\leqslant\mathbb{E}\int_{\mathbb{R}}u_{0x}^3\,dx+\left(6\mathbb{E}\sup_{t\in[0,T]}\|u\|_{H^1}^2+\frac{9}{4}\mathbb{E}\sup_{t\in[0,T]}\|u\|_{H^1}^4+3\mathbb{E}\sup_{t\in[0,T]}\|\Phi\|_{L_2^{0,-1}}\|u\|_{H^1}\right)t\\
&\quad\lesssim\mathbb{E}\int_{\mathbb{R}}u_{0x}^3\,dx+C\left(\mathbb{E}\|u_0\|_{H^1},T,\|\Phi\|_{L_2^{0,0}}\right)t.
\end{aligned}\tag{5.4}$$

Let $b=C(\mathbb{E}\|u_0\|_{H^1},T,\|\Phi\|_{L_2^{0,0}})$, when $t<t_0=\min\{T,-\mathbb{E}\int_{\mathbb{R}}u_{0x}^3\,dx/(2b)\}$ the inequality

$$\mathbb{E}\int_{\mathbb{R}}u_x^3\,dx+\frac{3}{2}\int_0^t\frac{1}{\mathbb{E}\|u\|_{H^1}^2}\left(\mathbb{E}\int_{\mathbb{R}}u_x^3\,dx\right)^2ds\lesssim\frac{1}{2}\mathbb{E}\int_{\mathbb{R}}u_{0x}^3\,dx\tag{5.5}$$

holds, which leads to the estimate

$$\mathbb{E}\int_{\mathbb{R}} u_x^3\,dx \lesssim \frac{\frac{1}{2}\mathbb{E}\int_{\mathbb{R}} u_{0x}^3\,dx}{1+\frac{3}{4}\frac{\int_{\mathbb{R}} u_{0x}^3\,dx\,t}{\mathbb{E}\|u\|_{H^1}^2}} \lesssim \frac{\frac{1}{2}\mathbb{E}\int_{\mathbb{R}} u_{0x}^3\,dx}{1+\frac{3}{4}\frac{\mathbb{E}\int_{\mathbb{R}} u_{0x}^3\,dx}{\mathbb{E}\|u_0\|_{H^1}^2+T\|\Phi\|_{L_2^{0,0}}}t} < 0, \tag{5.6}$$

for any $t < \min\{t_0, -\frac{4}{3\mathbb{E}\int_{\mathbb{R}} u_{0x}^3\,dx}(\mathbb{E}\|u_0\|_{H^1}^2 + T\|\Phi\|_{L_2^{0,0}})\}$. This implies that

$$t_0 \leqslant t_1 = -\frac{4}{3\mathbb{E}\int_{\mathbb{R}} u_{0x}^3\,dx}\big(\mathbb{E}\|u_0\|_{H^1}^2 + T\|\Phi\|_{L_2^{0,0}}\big). \tag{5.7}$$

Because if $t_0 > t_1 = -\frac{4}{3\mathbb{E}\|u_{0x}\|_{L^3}^3}(\mathbb{E}\|u_0\|_{H^1}^2 + T\|\Phi\|_{L_2^{0,0}})$, then

$$\lim_{t\to t_1}\mathbb{E}\int_{\mathbb{R}} u_x^3\,dx \lesssim \lim_{t\to t_1}\frac{\frac{1}{2}\mathbb{E}\int_{\mathbb{R}} u_{0x}^3\,dx}{1+\frac{3}{4}\frac{\mathbb{E}\int_{\mathbb{R}} u_{0x}^3\,dx\,t}{\mathbb{E}\|u_0\|_{H^1}^2+T\|\Phi\|_{L_2^{0,0}}}} = -\infty, \tag{5.8}$$

and the inequalities

$$\left|\mathbb{E}\int_{\mathbb{R}} u_x^3\,dx\right| \lesssim \mathbb{E}\|u\|_{H^1}^2\|u\|_{H^q} \lesssim \mathbb{E}\|u\|_{H^q}, \tag{5.9}$$

where $q \in (3/2, \infty)$.

$$\mathbb{E}\|u\|_{H^q} \gtrsim \left|\mathbb{E}\int_{\mathbb{R}} u_x^3\,dx\right| \gtrsim \frac{-\frac{1}{2}\mathbb{E}\int_{\mathbb{R}} u_{0x}^3\,dx}{1+\frac{3}{4}\frac{\mathbb{E}\int_{\mathbb{R}} u_{0x}^3\,dx\,t}{\mathbb{E}\|u_0\|_{H^1}^2+T\|\Phi\|_{L_2^{0,0}}}}. \tag{5.10}$$

This means that the H^q-norm with $q \in (3/2, \infty)$ of the solution u blows up at the time $t = t_1 < T$.

On the other hand, since

$$t_0 \leqslant t_1 = -\frac{4}{3\mathbb{E}\int_{\mathbb{R}} u_{0x}^3\,dx}\big(\mathbb{E}\|u_0\|_{H^1}^2 + T\|\Phi\|_{L_2^{0,0}}\big) \leqslant -\mathbb{E}\int_{\mathbb{R}} u_{0x}^3\,dx/(2b),$$

it follows $t_0 = T$, which combined with (5.6), (5.9) and (4.3) shows $\lim_{t\to T}\mathbb{E}\|u\|_{H^q} = \infty$.

To verify $\lim_{t\to T}\sup\mathbb{E}\|u_x\|_{L^\infty} = \infty$, by Lemma 4.4(2), it follows

$$\mathbb{E}\sup_{t\in[0,T]}\|u\|_{H^q}^2 \lesssim \mathbb{E}\|u_0\|_{H^{q-1}}^2 + \mathbb{E}\|u_0\|_{H^q}^2 + T\|\Phi\|_{L_2^{0,q-1}}^2 + \mathbb{E}\int_0^T \|u_x\|_{L^\infty}\|u\|_{H^q}^2\,ds,$$

for any $q \in (3/2, s]$. It follows from Gronwall's inequality that

$$\mathbb{E}\sup_{t\in[0,T]}\|u\|_{H^q}^2 \lesssim \exp\left(\mathbb{E}\int_0^T \|u_x\|_{L^\infty}\,ds\right). \tag{5.11}$$

Therefore, if $\lim_{t\to T}\sup\mathbb{E}\|u_x\|_{L^\infty}<\infty$, then it would lead to $\lim_{t\to T}\sup\mathbb{E}\|u\|_{H^q}<\infty$, which is contrary to $\lim_{t\to T}\sup\mathbb{E}\|u\|_{H^q}=\infty$. Hence, $T=T^*$ is the finite time for u and u ceases existing in L^∞ and H^q for any $q\in(3/2,s]$, respectively. $\square$

Corollary 5.1. *Suppose $\Phi\in L_2^{0,s-1}$, $u_0\in H^s(\mathbb{R})$ for a.e. $\omega\in\Omega$ is $\mathcal{F}_0$-measurable for $s>3/2$ and that T_0 is the maximum time for the solution u of* (1.1)–(1.2) *to exist in the space $L^2(\Omega;C([0,T_0);H^s(\mathbb{R})))$. If $T_0<\infty$, then $\sup_{0\leqslant t<T_0}\mathbb{E}\|u_x\|_{L^\infty}=\infty$.*

Proof. Assume that $T_0<\infty$ and $\sup_{0\leqslant t<T_0}\mathbb{E}\|u_x\|_{L^\infty}<\infty$. Then it follows from (5.9) that $\sup_{0\leqslant t<T_0}\mathbb{E}\|u_x\|_{L^\infty}<\infty$ for any $q\in(3/2,s]$. Hence, one may use an argument similar to that in the proof of Theorem 4.1 to show that u has a unique extension as a solution of (1.1)–(1.2) in the space $L^2(\Omega;C([0,T_1);H^s(\mathbb{R})))$ for some $T_1>T_0$ which contradicts the condition that T_0 is maximum. Hence, $T_0=\infty$. $\square$

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References

[1] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations, II: The KdV-equation, Geom. Funct. Anal. 3 (1993) 209–262.
[2] J.L. Bona, R. Smith, The initial value problem for the Korteweg–de Vries equation, Philos. Trans. R. Soc. Lond. Ser. A 278 (1975) 555–601.
[3] J.L. Bona, Y.A. Li, Decay and analyticity of solitary waves, J. Math. Pures Appl. 76 (1997).
[4] Z. Brzezniak, B. Zdzislowshi, J. Seidler, Stochastic nonlinear beam equations, Probab. Theory Related Fields 132 (2005) 119–149.
[5] P.J. Byers, The initial value problem for a KdV-type equation and related bilinear estimate, Dissertation, University of Notre Dame, 2003.
[6] A. Constantin, On the Cauchy problem for the periodic Camassa–Holm equation, J. Differential Equations 141 (2) (1997) 218–235.
[7] A. Constantin, J. Escher, Global weak solutions for a shallow water equation, Indiana Univ. Math. J. 47 (4) (1998) 1527–1545.
[8] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Global well-posedness for the KdV in Sobolev spaces of negative index, Electron. J. Differential Equations 26 (2001) 1–7.
[9] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Global well-posedness for Schrödinger equations with derivative, SIAM J. Math. Anal. 33 (2001) 649–666.
[10] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, Sharp global well-posedness for KdV and modified KdV on **R** and **T**, J. Amer. Math. Soc. 16 (2003) 705–749.
[11] R. Camassa, D. Holm, An integrable shallow water equation with peaked solitons, Phys. Rev. Lett. 71 (1993) 1661–1664.
[12] A. de Bouard, A. Debussche, Y. Tsutsumi, White noise driven Korteweg–de Vries equation, J. Funct. Anal. 169 (1999) 532–558.
[13] A. de Bouard, A. Debussche, On the stochastic Korteweg–de Vries equation, J. Funct. Anal. 154 (1998) 215–551.
[14] A. de Bouard, A. Debussche, The stochastic nonlinear Schrödinger equation in H^1, Stoch. Anal. Appl. 21 (2003) 97–126.
[15] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, Cambridge University Press, Cambridge, 1992.
[16] B. Fuchssteiner, A. Fokas, Symplectic structures, their Bäklund transformations and hereditary symmetries, Phys. D 4 (1) (1981) 47–66.
[17] J. Ginibre, Y. Tsutsumi, Uniqueness of solutions for the generalized Korteweg–de Vries equation, SIAM J. Math. Anal. 20 (6) (1989) 1388–1425.
[18] A. Himonas, G. Misiolek, The Cauchy problem for a shallow water type equation, Comm. Partial Differential Equations 23 (1998) 123–139.
[19] A. Himonas, G. Misiolek, Well-posedness of the Cauchy problem for a shallow water equation on the circle, J. Differential Equations 161 (2000) 479–495.
[20] A. Himonas, G. Misiolek, The initial value problem for a fifth order shallow water, in: Analysis, Geometry, Number Theory: The Mathematics of Leon Ehrenpreis, in: Contemp. Math., vol. 251, Amer. Math. Soc., Providence, RI, 2000, pp. 309–320.

[21] C.E. Kenig, G. Ponce, L. Vega, The Cauchy problem for the Korteweg–de Vries equation in Sobolev spaces of negative indices, Duke Math. J. 71 (1993) 1–21.
[22] C.E. Kenig, G. Ponce, L. Vega, A bilinear estimate with applications to the KdV equation, J. Amer. Math. Soc. 9 (1996) 573–603.
[23] T. Kato, G. Ponce, Commutator estimates and the Euler and Navier–Stokes equation, Comm. Pure Appl. Math. 41 (1988) 891–907.
[24] Y.A. Li, P.J. Olver, Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation, J. Differential Equations 162 (1) (2000) 27–63.
[25] X. Liu, Y. Jin, The Cauchy problem of a shallow water equation, Acta Math. Sin. (Engl. Ser.) 30 (2004) 1–16.
[26] G. Misiolek, Classical solutions of the periodic Camassa–Holm equation, Geom. Funct. Anal. 12 (5) (2002) 108–1104.
[27] G. Rodríguez-Blanco, On the Cauchy problem for the Camassa–Holm equation, Nonlinear Anal. 46 (3) (2001) 309–327.
[28] C.F. Sun, H.J. Gao, J.Q. Duan, B. Schmalfuss, Rare events in the Boussinesq system with fluctuating dynamical boundary conditions, J. Differential Equations 248 (2010) 1269–1296.
[29] T. Tao, Multilinear weighted convolution of L^2 functions, and applications to nonlinear dispersive equation, Amer. J. Math. 123 (2001) 839–908.
[30] H. Wang, S.B. Cui, Global well-posedness of the Cauchy problem of the fifth-order shallow water equation, J. Differential Equations 230 (2006) 600–613.
[31] G.L. Wang, B.L. Guo, Well-posedness of stochastic Korteweg–de Vries–Benjamin–Ono equation, Front. Math. China 5 (1) (2010) 161–177.
[32] Z.P. Xin, P. Zhang, On the weak solutions to a shallow water equation, Comm. Pure Appl. Math. 53 (11) (2000) 1411–1433.
[33] Z.P. Xin, P. Zhang, On the uniqueness and large time behavior of the weak solutions to a shallow water equation, Comm. Partial Differential Equations 27 (9–10) (2002) 1815–1844.
[34] X.Y. Yang, Y.S. Li, Global well-posedness for a fifth-order shallow water equation in Sobolev spaces, J. Differential Equations 248 (2010) 1458–1472.